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Causal trees and timed causal trees categorically*

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Abstract. Causal trees represented by Darondeau and Degano are one of the truly concurrent model for distributed systems and processes. The model is more basic than other truly concurrent models because it defines concurrency and causality with respect to a branch, but on the other hand it is more expressive than the latter because its possible runs can be freely specified in terms of a tree.

The intention of the paper is to connect two distinct approaches to which the category theory has been applied in order to investigate causal trees and their timed extensions. In particular, we establish that in the case of causal trees, the approaches are equivalent, but it is not true in the case of timed causal trees.

Keywords: timed causal trees, category theory, syntax and semantics.

1. Introduction

In [13], Winskel used the category theory to relate and unify the many concurrency models. The main idea is to formalize models as categories: each model is equipped with a notion of morphism. Theoretical category notions such as adjunctions and (co)reflections can then be applied to clarify the relationships between the models. This approach has helped to clarify the connections between interleaving and truly concurrent models such as synchronization trees, transition systems, event structures, trace languages, asynchronous transition systems, Petri nets, causal trees and others (for instance, see [1, 14, 5]). Later, in [8], the authors establish connections between some real-time extensions of the concurrency models. This approach has several advantages. First, the structure in the category of models (for example, products and coproducts, see [2]) allows us to construct composite models from simpler sub-components. Second, the categorical approach often makes it straightforward to extend and generalize results by modifying the structures under consideration. Finally, the approach has also been applied to unify and understand apparent differences between the extensive amount of research within the field of behavioral equivalences. For example, in [7] Joyal, Nielsen, and Winskel have proposed abstract ways of capturing the notion of behavioral equivalence through open maps based bisimilarity and its logical counterpart — path bisimilarity. As shown in [7, 10, 6, 11, 12], bisimilarity induced by open maps makes possible a uniform definition of the

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numerous suggested behavioral equivalences (e.g., trace and testing equivalences, bisimulation, barbed and weak bisimulations, strong history preserving bisimulation, etc.) across a wide range of models for concurrency (e.g., transition systems, event structures, Petri nets, timed transition systems, timed event structures, etc.).

In [9], Meseguer and Montanari introduced another categorical approach in the setting of the model of Petri nets. According to it, each Petri net can be regarded as a labelled transition system for its operational semantics, in which the states are markings and the transitions are sequential and parallel compositions of the events of the net. In fact, such transition system yields a category called the behavior category of the Petri net. Then the authors defined a semantic category of Petri nets, whose objects are their behavior categories.

In [3], Brown and Gurr established a precise connection between the Winskel's category of Petri Nets (in which morphisms consider only a static structure) and the semantic category of Petri Nets (in which morphisms consider a purely dynamic behavior of nets) and applied these results to the Petri Nets timing.

This paper connects the two mentioned above approaches in the setting of another truly concurrent model — causal trees of Darondeau and Degano [4] — and its timed variant — timed causal trees. In particular, we establish that in the case of causal trees the approaches are equivalent, but it is not true in the case of timed causal trees.

The rest of the paper is organized as follows. The basic notions and notations related to causal trees and definitions of the structural (whose objects are causal trees) and semantic (whose objects are behavior categories) categories of causal trees ($\mathbb{CT}$ and $\mathbf{CT}$, resp.) are introduced in Section 2. In the next section, we prove that the categories mentioned above are equivalent. The timed extension of causal trees and two categories of timed causal trees are represented in Section 4. In Section 5, we establish a coreflection between the semantic category whose objects are behavior categories with timings and the structural category of timed causal trees. Section 6 contains conclusion and some remarks on future work.

2. Causal trees and their categories

The causal trees were introduced by Darondeau and Degano in the late 1980s [4]. These trees are a special kind of synchronization trees with enriched action labels that supply information about the transitions that are causally dependent on each other.

Recall some elementary definitions of causal trees theory. For a start, we introduce some auxiliary notions and notations. Let L be a finite alphabet of actions. Let us consider a definition of the concept of synchronization trees.

Definition 1. *A* synchronization tree $\mathcal{S}$ *is a tuple* (S, s_{in}, L, T)*, where* S *is a set of states with the initial state* s_{in} *and* $T \subseteq S \times L \times S$ *is a set of transitions, such that*

(i) for all states $s \in S$*, there exists a sequence* $s_{in} \xrightarrow{a_1} s_1 \ldots s_{k-1} \xrightarrow{a_k} s_k$ $(k \geq 0)$ *such that* $s = s_k$*,*

(ii) for all sequences $s_0 \xrightarrow{a_1} s_1 \ldots s_{k-1} \xrightarrow{a_k} s_k$ $(k \geq 0)$*, if* $s_0 = s_k$ *then* $k = 0$*,*

(iii) if $s' \xrightarrow{a} s$ *and* $s'' \xrightarrow{a'} s$*, then* $s' = s''$ *and* $a = a'$*.*

We shall write $s \xrightarrow{a} s'$ *to denote a transition* (s, a, s')*.*

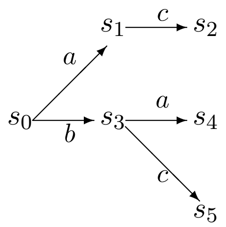
Thus, we have that in a synchronization tree every state is reachable and there is no backwards branching or cycles. Moreover, it holds that in a synchronization tree $\mathcal{S}$ for all states $s \in S$ there exists a unique sequence $s_{in} \xrightarrow{a_1} s_1 \ldots s_{k-1} \xrightarrow{a_k} s_k$ $(k \geq 0)$ such that $s = s_k$, due to the items (i) and (iii). We denote the sequence as r_s.

Now, we can recall the definition of causal trees.

Definition 2. *A* causal tree $\mathcal{C}$ *over* L *is a tuple* $(S, s_{in}, L, T, <)$*, where* $(S,\ s_{in},\ L,\ T)$ *is a synchronization tree over* L *and* $< \subseteq T \times T$*,* the causal dependency relation*, is a strict order such that if* $(s, \sigma, s') < (s'', \sigma', s''')$*, then there exists a sequence* $s' = s_0 \xrightarrow{\sigma_1} s_1 \ldots s_{k-1} \xrightarrow{\sigma_k} s_k = s''$ $(k \geq 0)$*.*

Intuitively, this condition reflects a natural property of causality: if a transition is the cause of another transition, then the first transition must have happened before the second one. We say that two transitions (s, σ, s'), $(u, \sigma', u') \in T$ are *consistent* (denoted $(s, \sigma, s')\ Con\ (u, \sigma', u')$) iff they appear on the same branch. Also, we say that two consistent transitions (s, σ, s') and (s'', σ', s''') are concurrent iff they are not identical and not related by $<$. Note that, in contrast to event structures, here concurrency is meaningful only when interpreted with respect to a branch.

Example 1. *An example of a causal tree,* $\mathcal{C}_1$*, is depicted in Figure 1. This causal tree consists of six states (*s_0*,* s_1*,* s_2*,* s_3*,* s_4 *and* s_5*) and five transitions (*(s_0, a, s_1)*,* (s_1, c, s_2)*,* (s_0, b, s_3)*,* (s_3, a, s_4) *and* (s_4, c, s_5)*). Moreover, the first transition is the cause for the second one and the third transition is the cause for the fifth one. It is clear, that* (s_0, b, s_3) *and* (s_3, a, s_4) *are concurrent transitions, but* (s_0, b, s_3) *and* (s_1, c, s_2) *are not.*



$\mathcal{C}_1$: $(s_0, a, s_1) < (s_1, c, s_2)$; $(s_0, b, s_3) < (s_3, c, s_5)$.

Figure 1. An example of a causal tree

In 2007, Fröshle and Lasota [5] define the structural category of causal trees and investigated how the category relates to other model categories. Now, we are ready to introduce this category. Note that the morphisms of the category preserve concurrency as the morphisms of other truly concurrent models.

Lemma 1. *The following data:*

- *objects are causal trees over* L,
- *morphisms from an object* $\mathcal{C} = (S,\ s_{in},\ L,\ T,\ <)$ *to an object* $\mathcal{C}' = (S',\ s'_{in},\ L,\ T',\ <')$ *are functions* $f : S \to S'$ *such that such that* $f(s_{in}) = s'_{in}$, *for all* $(s, \sigma, s') \in T$ *it holds that* $(f(s), \sigma, f(s')) \in T'$, *and if* $(f(s), \sigma, f(s')) <' (f(u), \sigma', f(u'))$ *then* $(s, \sigma, s') < (u, \sigma', u')$, *for all* (s, σ, s') Con (u, σ', u'),
- *composition is given by function composition,*

define a structural category $\mathbb{CT}$ *of causal trees over* L.

Define the behavior of causal trees. A *run* of $\mathcal{C}$ is a sequence $r := s_{in} = s_0 \underset{K_1}{\overset{\sigma_1}{\to}} s_1 \ldots s_{n-1} \underset{K_n}{\overset{\sigma_n}{\to}} s_n$ $(n \geq 0)$, where $K_i = \{j \mid 1 \leq j < i,\ (s_{j-1}, \sigma_j, s_j) < (s_{i-1}, \sigma_i, s_i)\}$. We identify the state s_n of the run r as $last(r)$. The run r as above is said to *generate the word* $\alpha(r) = \sigma_1 \ldots \sigma_n$ and the *causal sequence* $cas(r) = K_1 \ldots K_n$. The set of runs of $\mathcal{C}$ is denoted as $Runs(\mathcal{C})$.

Example 2. *Contemplate the causal tree* $\mathcal{C}_1$ *in Figure 1. It holds, that* $r := s_0 \underset{\emptyset}{\overset{a}{\to}} s_1 \underset{\{1\}}{\overset{c}{\to}} s_2$ *is a run of* $\mathcal{C}_1$. *Moreover,* r *generates the word* **ac** *and the causal sequence* $\emptyset\{\mathbf{1}\}$.

Now we define the behavior of a causal tree $\mathcal{C}$ by the category with the runs of $\mathcal{C}$ as objects and the extensions of the runs as morphisms.

Lemma 2. *Let $\mathcal{C} = (S, s_{in}, L, T, <)$ be a causal tree over L. The following data:*

- *objects are the runs of $\mathcal{C}$,*
- *morphisms are run extensions, i.e. $\mu : r \to r'$ is a morphism from a run r to a run r' if and only if r is a prefix of r',*
- *composition of morphisms is given by composition of functions,*

define a category $\mathbf{Beh}(\mathcal{C})$ *(the* behavior category *of $\mathcal{C}$).*

We expect a map between behavior categories to indicate how to simulate, or implement, the runs of one causal tree in another. An evident choice of a map between behavior categories is a structure-preserving functor.

Lemma 3. *The following data:*

- *objects are the behavior categories generated by causal trees over L,*
- *morphisms are functors $\lambda : \mathbf{Beh}(\mathcal{C}) \to \mathbf{Beh}(\mathcal{C}')$ (i.e. mappings that*
  - *(i) associate to each object r of $\mathbf{Beh}(\mathcal{C})$ an object $\lambda(r)$ of $\mathbf{Beh}(\mathcal{C}')$,*
  - *(ii) associate to each morphism $\mu : r \to r'$ of $\mathbf{Beh}(\mathcal{C})$ a morphism $\lambda(\mu) : \lambda(r) \to \lambda(r')$ of $\mathbf{Beh}(\mathcal{C}')$ and preserve identity morphisms and the composition of morphisms)*

  such that for all objects r of $\mathbf{Beh}(\mathcal{C})$ with $cas(r) = K_1 \dots K_n$, $\alpha(\lambda(r)) = \alpha(r)$ and $cas(\lambda(r)) = K'_1 \dots K'_n$ with $K'_i \subseteq K_i$ $(1 \le i \le n)$,
- *composition is given by functor composition,*

define a semantic category $\mathbf{CT}$ *of causal trees over L.*

3. Relating the categories $\mathbb{CT}$ and CT

In this section, we will obtain a natural connection between a category whose objects are behavior categories and a category whose objects are causal trees. First, let us establish the following useful fact.

Lemma 4. *Given a morphism $f : \mathcal{C} \to \mathcal{C}'$ of $\mathbb{CT}$, there is a functor $\bar{f} : \mathbf{Beh}(\mathcal{C}) \to \mathbf{Beh}(\mathcal{C}')$ which takes an object $r = s_{in} \overset{\sigma_1}{\underset{K_1}{\to}} s_1 \dots s_{n-1} \overset{\sigma_n}{\underset{K_n}{\to}} s_n$ of $\mathbf{Beh}(\mathcal{C})$ to the object $\bar{f}(r) = f(s_{in}) \overset{\sigma_1}{\underset{K'_1}{\to}} f(s_1) \dots f(s_{n-1}) \overset{\sigma_n}{\underset{K'_n}{\to}} f(s_n)$ of $\mathbf{Beh}(\mathcal{C}')$ with $K'_i = \{j \mid 1 \le j < i,\ (f(s_{j-1}), \sigma_j, f(s_j)) <_{\mathcal{C}'} (f(s_{i-1}), \sigma_i, f(s_i))\}$ $(1 \le i \le n)$, and a morphism $m : r \to r'$ of $\mathbf{Beh}(\mathcal{C})$ to the unique morphism $\bar{f}(m) : \bar{f}(r) \to \bar{f}(r')$ of $\mathbf{Beh}(\mathcal{C}')$. Moreover, $\bar{f}$ is a morphism of $\mathbf{CT}$.*

Proof. W.l.o.g. assume $\mathcal{C} = (S, s_{in}, L, T, <)$, $\mathcal{C}' = (S', s'_{in}, L, T', <')$, and $f : \mathcal{C} \to \mathcal{C}'$ is a morphism of $\mathbb{CT}$. Take an arbitrary object r of $\mathbf{Beh}(\mathcal{C})$ with $cas(r) = K_1 \dots K_n$. Clearly, $\bar{f}(r)$ is an object of $\mathbf{Beh}(\mathcal{C}')$ with $\alpha(\bar{f}(r)) = \alpha(r)$ and $cas(\bar{f}(r)) = K'_1 \dots K'_n$, where

$$K'_i = \{j \mid 1 \leq j < i,\ (f(s_{j-1}), \sigma_j, f(s_j)) <' (f(s_{i-1}), \sigma_i, f(s_i))\}(1 \leq i \leq n),$$

because f is a morphism of $\mathbb{CT}$. Next, since $K_i = \{j \mid 1 \leq j < i,\ (s_{j-1}, \sigma_j, s_j) < (s_{i-1}, \sigma_i, s_i)\}(1 \leq i \leq n)$ and f is a morphism of $\mathbb{CT}$, we may conclude that $K'_i \subseteq K_i$ $(1 \leq i \leq n)$.

Moreover, since in a behavior category there is at most one morphism between any two objects, we have that for each morphism $\mu : r \to r'$ of $\mathbf{Beh}(\mathcal{C})$ there is a unique morphism $\bar{f}(\mu) : \bar{f}(r) \to \bar{f}(r')$ of $\mathbf{Beh}(\mathcal{C}')$, $\bar{f}(id_r) = id_{\bar{f}(r)}$, and $\bar{f}(\mu_2 \circ \mu_1) = \bar{f}(\mu_2) \circ \bar{f}(\mu_1)$ for all morphisms $\mu_1 : r_1 \to r_2$ and $\mu_2 : r_2 \to r_3$ of $\mathbf{Beh}(\mathcal{C})$. Thus, $\bar{f} : \mathbf{Beh}(\mathcal{C}) \to \mathbf{Beh}(\mathcal{C}')$ is indeed a morphism of $\mathbf{CT}$.

The result enables us to define a functor between $\mathbb{CT}$ and $\mathbf{CT}$.

Definition 3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be causal trees and $f : \mathcal{C} \to \mathcal{C}'$ be a morphism of $\mathbb{CT}$ between them. Define a functor $\mathfrak{ct2beh}$, which takes a causal tree $\mathcal{C}$ to its behavior category $\mathbf{Beh}(\mathcal{C})$ and a morphism f to the functor $\bar{f}$.*

Theorem 1. $\mathfrak{ct2beh}$ *is a fully faithful and essentially surjective functor.*

Proof. First, it is clear that $\mathfrak{ct2beh}$ is indeed a functor by Lemma 4.

Second, we need to show that $\mathfrak{ct2beh}$ is a fully faithful functor. Take an arbitrary pair of objects $\mathcal{C}$ and $\mathcal{C}'$ of $\mathbb{CT}$. Define the function

$$F_{\mathcal{C},\mathcal{C}'} : Hom_{\mathbb{CT}}(\mathcal{C}, \mathcal{C}') \to Hom_{\mathbf{CT}}(\mathbf{Beh}(\mathcal{C}), \mathbf{Beh}(\mathcal{C}'))$$

as follows: $F_{\mathcal{C},\mathcal{C}'}(\mu) = \mathfrak{ct2beh}(\mu) = \bar{\mu}$ for all morphisms $\mu : \mathcal{C} \to \mathcal{C}'$ of $\mathbb{CT}$. Clearly, $F_{\mathcal{C},\mathcal{C}'}$ is indeed a function, because $\mathfrak{ct2beh}$ is a functor. Check that $F_{\mathcal{C},\mathcal{C}'}$ is bijective.

Take an arbitrary morphism $g : \mathbf{Beh}(\mathcal{C}) \to \mathbf{Beh}(\mathcal{C}')$ of $\mathbf{CT}$. Define a mapping $f : S \to S'$ as follows: for all $s \in S$, $f(s) = last(g(r_s))$, where r_s is run of $\mathcal{C}$ such that $last(r_s) = s$. Since $\mathcal{C}$ is a causal tree, we have that for all $s \in S$ there is the only run r_s such that $last(r_s) = s$. This implies that $f : S \to S'$ is a function. It is routine to check that f is a morphism of $\mathbb{CT}$ from $\mathcal{C}$ to $\mathcal{C}'$. Moreover, $F_{\mathcal{C},\mathcal{C}'}(f) = \mathfrak{ct2beh}(f) = \bar{f} = g$. Hence, $F_{\mathcal{C},\mathcal{C}'}$ is surjective, i.e. $\mathfrak{ct2beh}$ is a full functor.

Take two arbitrary morphisms $\mu : \mathcal{C} \to \mathcal{C}'$ and $\mu' : \mathcal{C} \to \mathcal{C}'$ such that $F_{\mathcal{C},\mathcal{C}'}(\mu) = F_{\mathcal{C},\mathcal{C}'}(\mu')$. This implies that $\bar{\mu} = \bar{\mu'}$. Check that $\mu(s) = \mu'(s)$ for all $s \in S$. Take an arbitrary $s \in S$ with a run r_s such that $last(r_s) = s$. Since $\bar{\mu} = \bar{\mu'}$, we have that $last(\bar{\mu}(r_s)) = last(\bar{\mu'}(r_s))$. Clearly, $last(\bar{\mu}(r_s))$

$= \mu(s)$ and $last(\bar{\mu}'(r_s)) = \mu'(s)$. Furthermore, $\mu(s) = \mu'(s)$. Hence, $F_{\mathcal{C},\mathcal{C}'}$ is injective, i.e. $\mathfrak{ct2beh}$ is a faithful functor.

Third, we should prove that $\mathfrak{ct2beh}$ is an essentially surjective functor. By the definition of functor $\mathfrak{ct2beh}$, it is easy to see that each object $\mathbf{Beh}(\mathcal{C})$ of $\mathbf{CT}$ is equal to an object of the form $\mathfrak{ct2beh}(\mathcal{C})$ for some object $\mathcal{C}$ of $\mathbb{CT}$. Hence, $\mathfrak{ct2beh}$ is essentially surjective.

Corollary 1. *The functor* $\mathfrak{ct2beh}$ *yields an equivalence of categories* $\mathbb{CT}$ *and* $\mathbf{CT}$.

Corollary 2. *The functor* $\mathfrak{ct2beh}$ *has a fully faithful right and left adjoint functor* $\mathfrak{G}$*, which maps each behavior category* $\mathbf{Beh}(\mathcal{C})$ *to the causal tree* $\mathcal{C}_{beh}$ *of the form:*

$$(Objects(\mathbf{Beh}(\mathcal{C})), r_0, L, Tran, <_{beh}),$$

where $Objects(\mathbf{Beh}(\mathcal{C}))$ *is a set of objects of* $\mathbf{Beh}(\mathcal{C})$ *(or runs of* $\mathcal{C}$*),* r_0 *is the initial object of* $\mathbf{Beh}(\mathcal{C})$ *(or the initial run of* $\mathcal{C}$*),* $(r_{n-1}, a_n, r_n) \in Tran$ *if and only if there is a morphism* $m : r_{n-1} \to r_n$ *with* $\alpha(r_n) = \alpha(r_{n-1})a_n$*, and* $(r_{n-1}, a_n, r_n) <_{beh} (r_{m-1}, a_m, r_m)$ *if and only if* $n < m$*, there is a morphism* $m : r_n \to r_{m-1}$*,* $cas(r_m) = K_1 \dots K_m$ *and* $n \in K_m$*;*
and each morphism $\lambda : \mathbf{Beh}(\mathcal{C}) \to \mathbf{Beh}(\mathcal{C}')$ *of* $\mathbf{CT}$ *to the morphism* $\lambda : \mathcal{C}_{beh} \to \mathcal{C}'_{beh}$ *of* $\mathbb{CT}$.

Moreover, the unit η *and the counit* ε *of the adjunction are (natural) isomorphisms.*

Thus, we established that the categories $\mathbf{CT}$ and $\mathbb{CT}$ are equivalent.

4. Timed causal trees

A *time object* $\Im$ is a tuple $(\mathbb{T}, \leq^+, 0, \bigoplus)$, where $(\mathbb{T}, \leq^+, 0)$ is a partially order set with bottom element 0, and $(\mathbb{T}, \bigoplus, 0)$ is a commutative monoid such that $\bigoplus$ is monotone in each argument with respect to $\leq^+$, and $x \leq^+ x \bigoplus y$, $x =^+ x \bigoplus x$ and if $x \leq^+ y$ then $x \bigoplus y =^+ y$, for all $x, y \in \mathbb{T}$.

$\Im$ shall also denote the underlying category $(\mathbb{T}, \leq^+)$ of the time object.

Example 3. *The following are examples of time objects:*

- $\mathbb{T}$ *is a set of natural (*$\mathbf{N}$*) or real (*$\mathbf{R}_+$*) numbers with the usual ordering, and* $\bigoplus$ *is* max.
- $\mathbb{T}$ *is a set of intervals of real numbers (*$Int(\mathbf{R}_+)$*) with the following ordering:* $[a, b] \leq [c, d] \iff a \leq c$ *and* $b \leq d$*; and* $[a, b] \bigoplus [c, d] := [\max\{a, c\}, \max\{b, d\}]$.

Definition 4. *Let* $\Im = (\mathbb{T}, \leq^+, 0, \bigoplus)$ *be a time object. A* $\mathbb{T}$-timed causal tree over *L is a pair* $(\mathcal{C}, \tau)$*, where* $\mathcal{C}$ *is a causal tree over* L *and* $\tau : T \to \mathbb{T}$ *is a timed function such that if* $(s, \sigma, s') < (s'', \sigma', s''')$ *then* $\tau((s, \sigma, s')) \leq^+ \tau((s'', \sigma', s'''))$.

Informally, a timed function assigns to each transition of a tree the earliest time moment of running in the case when $\mathbb{T}$ is a set of natural or real numbers, and the time interval of running in the case when $\mathbb{T}$ is a set of intervals.

Example 4. *Consider the causal tree $\mathcal{C}_1$ in Figure 1 and the following two maps $\tau_1, \tau_2 : T_1 \to \mathbf{R}_+$, which map transitions to the time moments of running: $\tau_1((s_0, a, s_1)) = 1$, $\tau_1((s_1, c, s_2)) = 1$, $\tau_1((s_3, a, s_4)) = 1$, $\tau_1((s_0, b, s_3)) = 3$, $\tau_1((s_4, c, s_5)) = 4$; and $\tau_2((s_0, a, s_1)) = 5$, $\tau_2((s_1, c, s_2)) = 4$, $\tau_2((s_0, b, s_3)) = 3$, $\tau_2((s_3, a, s_4)) = 1$, $\tau_2((s_4, c, s_5)) = 2$. It is clear, that the pair $(\mathcal{C}_1, \tau_1)$ is a $\mathbb{T}$-timed causal tree, but the pair $(\mathcal{C}_1, \tau_2)$ is not, because $\tau_2((s_1, c, s_2)) = 4 \leq^+ \tau_2((s_0, a, s_1)) = 5$ despite the fact that $(s_0, a, s_1) < (s_1, c, s_2)$.*

Now, we define a category with objects $\mathbb{T}$-timed causal trees over L and morphisms that preserve the structure of $\mathbb{T}$-timed causal trees and are "faster than" mappings.

Lemma 5. *The following data:*

- *objects are $\mathbb{T}$-timed causal trees over L,*
- *morphisms from an object $(\mathcal{C}, \tau)$ to an object $(\mathcal{C}', \tau')$ are morphisms $f : \mathcal{C} \to \mathcal{C}'$ such that $\tau'((f(s), \sigma, f(s'))) \leq^+ \tau((s, \sigma, s'))$,*
- *composition is given by function composition,*

define a structural category $\mathbb{TCT}$ of $\mathbb{T}$-timed causal trees over L.

Thus, causal trees are intuitively those $\mathbb{T}$-timed causal trees without a timed function. Formally, the two models are related by a reflection. Consider an evident forgetful functor $\mathfrak{U}$ from $\mathbb{TCT}$ to $\mathbb{CT}$ mapping an object $(\mathcal{C}, \tau)$ of $\mathbb{TCT}$ to the object $\mathcal{C}$ of $\mathbb{CT}$ and a morphism $f : (\mathcal{C}, \tau) \to (\mathcal{C}', \tau')$ of $\mathbb{TCT}$ to the morphism $f : \mathcal{C} \to \mathcal{C}'$ of $\mathbb{CT}$. Now we show that $\mathfrak{U}$ has a right adjoint.

Theorem 2. *The forgetful functor $\mathfrak{U} : \mathbb{TCT} \to \mathbb{CT}$ has a right adjoint $\mathfrak{T} : \mathbb{CT} \to \mathbb{TCT}$. Moreover, the adjunction is a reflection, i.e. the counit is a (natural) isomorphism.*

Proof. Let $\mathcal{C}$ and $\mathcal{C}'$ be causal trees and $f : \mathcal{C} \to \mathcal{C}'$ be a morphism of $\mathbb{CT}$. We define $\mathfrak{T}(\mathcal{C}) = (\mathcal{C}, \tau_0)$, where τ_0 is the constant 0 function, and $\mathfrak{T}(f) = f$. It is routine to verify that $\mathfrak{T}$ is a functor.

Next, we have that $\mathfrak{U}(\mathfrak{T}(\mathcal{C})) = \mathcal{C}$. Clearly, $id_C : \mathfrak{U}(\mathfrak{T}(\mathcal{C})) = \mathcal{C} \to \mathcal{C}$ is a (iso)morphism of $\mathbb{CT}$.

Now we should prove that $(\mathfrak{T}(\mathcal{C}), id_C)$ is a coreflection of $\mathcal{C}$ along $\mathfrak{U}$, i.e. whenever $(\mathcal{C}', \tau')$ is a $\mathbb{T}$-timed causal tree and $f : \mathfrak{U}((\mathcal{C}', \tau')) \to \mathcal{C}$ is a morphism of $\mathbb{CT}$, there exists a unique morphism $g : (\mathcal{C}', \tau') \to \mathfrak{T}(\mathcal{C})$ such that $id_C \circ \mathfrak{U}(g) = f$. Since $\mathfrak{U}(g) = g$, we may conclude that g must be equal to f. The condition that $\tau_0 \circ g((s,a,s')) \leq^+ \tau'((s,a,s'))$ for all (s,a,s') is satisfied trivially.

Since $id_{C'} \circ \mathfrak{U}(\mathfrak{T}(f)) = f \circ id_C$ for all morphism $f : \mathcal{C} \to \mathcal{C}'$, it holds that $\mathfrak{T}$ is the right adjoint to $\mathfrak{U}$.

Finally, we may conclude that the counit ε is a natural isomorphism, since it associates each causal tree $\mathcal{C}$ with the isomorphism id_C.

Hence, $\mathbb{CT}$ embeds fully and faithfully into $\mathbb{TCT}$ and is equivalent to the full subcategory of $\mathbb{TCT}$ consisting of those $\mathbb{T}$-timed causal trees $(\mathcal{C}, \tau)$ that are isomorphic to $(\mathcal{C}, \tau_0)$.

We make a natural assumption that assigning a time to each transition of a causal tree determines a certain time of every run of the tree. The following lemma shows that we can use the behavior category to express this translating of times from transitions to runs.

Lemma 6. *Let $(\mathcal{C}, \tau)$ be a $\mathbb{T}$-timed causal tree over L. Then τ extends uniquely to a functor $\overline{\tau} : \mathbf{Beh}(\mathcal{C}) \to \Im$ such that $\overline{\tau}(r) = \bigoplus_{i=1}^{n} \tau((s_{i-1}, \sigma_i, s_i))$, for all object $r := s_{in} = s_0 \stackrel{\sigma_1}{\to} s_1 \ldots s_{n-1} \stackrel{\sigma_n}{\to} s_n$ of $\mathbf{Beh}(\mathcal{C})$. We shall call such a functor $\overline{\tau}$ a* $\mathbb{T}$-timing *of* $\mathbf{Beh}(\mathcal{C})$.

Example 5. *Take the $\mathbb{T}$-timed causal tree $(\mathcal{C}_1, \tau_1)$ from Example 4. It is easy to see, that the functor $\overline{\tau_1}$, defined as follows: $\overline{\tau_1}(r_{s_0}) = 0$, $\overline{\tau_1}(r_{s_1}) = 1$, $\overline{\tau_1}(r_{s_2}) = 1$, $\overline{\tau_1}(r_{s_3}) = 3$, $\overline{\tau_1}(r_{s_4}) = 3$, and $\overline{\tau_1}(r_{s_5}) = 4$ (where r_{s_i} is the unique run ending in s_i $(0 \leq i \leq 5)$), is a $\mathbb{T}$-timing of $\mathbf{Beh}(\mathcal{C}_1)$.*

Now we are ready to define a semantic category of timed causal trees.

Lemma 7. *The following data:*

- *objects are pairs $(\mathbf{Beh}(\mathcal{C}), \overline{\tau})$, where $\mathbf{Beh}(\mathcal{C})$ is the behavior category of a causal tree $\mathcal{C}$ over L and $\overline{\tau}$ is a $\mathbb{T}$-timing of $\mathbf{Beh}(\mathcal{C})$,*
- *morphisms from $(\mathbf{Beh}(\mathcal{C}), \overline{\tau})$ to $(\mathbf{Beh}(\mathcal{C}'), \overline{\tau}')$ are morphisms $\lambda : \mathbf{Beh}(\mathcal{C}) \to \mathbf{Beh}(\mathcal{C}')$ such that $\overline{\tau}' \circ \lambda(r) \leq^+ \overline{\tau}(r)$, for all objects r of $\mathbf{Beh}(\mathcal{C})$,*
- *composition is given by functor composition,*

define a semantic category **TCT** *of timed causal trees over L.*

Informally, in **TCT** a run r with the time t simulates another run r' with the time t' if r simulates r' in our original sense and furthermore $t \leq t'$. Thus, a timed specification is a specification of desired runs together with a limit on the execution time of each run.

Again, there is an evident forgetful functor $\mathfrak{UBeh}$ from **TCT** to **CT** mapping an object $(\mathbf{Beh}(\mathcal{C}), \overline{\tau})$ of **TCT** to the object $\mathbf{Beh}(\mathcal{C})$ of **CT** and a morphism $\lambda : (\mathbf{Beh}(\mathcal{C}), \overline{\tau}) \to (\mathbf{Beh}(\mathcal{C}'), \overline{\tau'})$ of **TCT** to the morphism $\lambda : \mathbf{Beh}(\mathcal{C}) \to \mathbf{Beh}(\mathcal{C}')$ of **CT**. Moreover, it is clear that the mapping $\mathfrak{TBeh}$ which takes the behavior category $\mathbf{Beh}(\mathcal{C})$ to the pair $(\mathbf{Beh}(\mathcal{E}), \overline{\tau_0})$ and the morphism $\lambda : \mathbf{Beh}(\mathcal{E}) \to \mathbf{Beh}(\mathcal{E}')$ to λ is a functor.

Theorem 3. *The forgetful functor* $\mathfrak{UBeh} : \mathbf{TCT} \to \mathbf{CT}$ *has a right adjoint* $\mathfrak{TBeh} : \mathbf{CT} \to \mathbf{TCT}$. *Moreover, the adjunction is a reflection, i.e. the counit is a (natural) isomorphism.*

Proof. The proof is similar to the proof of Theorem 2.

Thus, **CT** embeds fully and faithfully into **TCT** and is equivalent to the full subcategory of **TCT** consisting of the objects $(\mathbf{Beh}(\mathcal{C}), \overline{\tau})$ isomorphic to $(\mathbf{Beh}(\mathcal{C}), \overline{\tau_0})$.

5. Relating the categories $\mathbb{TCT}$ and TCT

In this section, we consider the relations between the timed extensions of the two equivalent categories $\mathbb{CT}$ and **CT**. First, we define a functor $\mathfrak{tct2beh} : \mathbb{TCT} \to \mathbf{TCT}$.

Definition 5. *Let* $(\mathcal{C}, \tau)$ *and* $(\mathcal{C}', \tau')$ *be* $\mathbb{T}$*-timed causal trees and* $\mu : (\mathcal{C}, \tau) \to (\mathcal{C}', \tau')$ *be a morphism of* $\mathbb{TCT}$ *between them. Define a functor* $\mathfrak{tct2beh}$ *which takes a* $\mathbb{T}$*-timed causal tree* $(\mathcal{C}, \tau)$ *to the pair* $(\mathbf{Beh}(\mathcal{C}), \overline{\tau})$ *and a morphism* μ *to the functor* $\bar{\mu}$.

Theorem 4. $\mathfrak{tct2beh}$ *is a faithful and essentially surjective functor.*

Proof. Using Theorem 1, it is routine to check that $\mathfrak{tct2beh}$ is a functor.

Take an arbitrary pair of objects $(\mathcal{C}, \tau)$ and $(\mathcal{C}', \tau')$ of $\mathbb{TCT}$. Define a function

$$F : Hom_{\mathbb{TCT}}((\mathcal{C}, \tau), (\mathcal{C}', \tau')) \to Hom_{\mathbf{TCT}}((\mathbf{Beh}(\mathcal{C}), \overline{\tau}), (\mathbf{Beh}(\mathcal{C}'), \overline{\tau'}))$$

such that $F(\mu) = \mathfrak{tct2beh}(\mu) = \mathfrak{ct2beh}(\mu)$ for all morphisms $\mu : (\mathcal{C}, \tau) \to (\mathcal{C}', \tau')$ of $\mathbb{TCT}$. Clearly, F is indeed a function, because $\mathfrak{tct2beh}$ is a functor. Next, take two arbitrary morphisms $\mu, \mu' : (\mathcal{C}, \tau) \to (\mathcal{C}', \tau')$ such that $F(\mu) = F(\mu')$. This implies that $\mathfrak{ct2beh}(\mu) = \mathfrak{ct2beh}(\mu')$. Since $\mathfrak{ct2beh}$ is a faithful functor, we may conclude that $\mu = \mu'$. Hence, F is injective, i.e. $\mathfrak{tct2beh}$ is a faithful functor.

Finally, it holds that each object $(\mathbf{Beh}(\mathcal{C}), \overline{\tau})$ of **TCT** is equal to an object of the form $\mathfrak{tct2beh}(\mathcal{C}, \tau)$ for some object $(\mathcal{C}, \tau)$ of $\mathbb{TCT}$. Thus, $\mathfrak{tct2beh}$ is essentially surjective.

Theorem 5. $\mathfrak{tct2beh}$ *is not a full functor.*

Proof. Contemplate the causal tree $\mathcal{C}_2$ with the empty causal dependency relation:

$$s_0 \xrightarrow{a} s_1 \xrightarrow{b} s_2,$$

and two timed functions $\tau, \tau' : T_2 \to \mathbb{T}$ such that $\tau(s_0, a, s_1) = 4$, $\tau'(s_0, a, s_1) = 4$, $\tau(s_1, b, s_2) = 1$ and $\tau'(s_1, b, s_2) = 3$. Since $\tau(s_1, b, s_2) < \tau'(s_1, b, s_2)$, we may conclude that $Hom_{\mathbb{TCT}}((\mathcal{C}_2, \tau), (\mathcal{C}_2, \tau')) = \emptyset$. On the other hand, it is clear that $\overline{\tau} = \overline{\tau'}$ and the identity functor belongs to the set of morphisms $Hom_{\mathbf{TCT}}((\mathbf{Beh}(\mathcal{C}_2), \overline{\tau}), (\mathbf{Beh}(\mathcal{C}_2), \overline{\tau'}))$. This implies that the last set is not empty. Hence, we get that any function H from $Hom_{\mathbb{TCT}}((\mathcal{C}_2, \tau), (\mathcal{C}_2, \tau'))$ to $Hom_{\mathbf{TCT}}((\mathbf{Beh}(\mathcal{C}_2), \overline{\tau}), (\mathbf{Beh}(\mathcal{C}_2), \overline{\tau'}))$ cannot be injective. Hence, $\mathfrak{tct2beh}$ is not full.

Corollary 3. *The categories* $\mathbb{TCT}$ *and* $\mathbf{TCT}$ *are not equivalent.*

Now we need to establish relationships between $\mathbb{TCT}$ and $\mathbf{TCT}$. To do that we show that $\mathfrak{tct2beh}$ has a left adjoint. First, specify a functor $\mathfrak{beh2tct} : \mathbf{TCT} \to \mathbb{TCT}$.

Definition 6. *Let* $(\mathbf{Beh}(\mathcal{C}), \overline{\tau})$ *and* $(\mathbf{Beh}(\mathcal{C}'), \overline{\tau'})$ *be the objects of* $\mathbf{TCT}$ *and* λ *be a morphism of* $\mathbf{TCT}$ *between them. Define* $\mathfrak{beh2tct}((\mathbf{Beh}(\mathcal{C}), \overline{\tau})) = (\mathfrak{G}(\mathbf{Beh}(\mathcal{C})), \overline{\tau}^*)$*, where* $\mathfrak{G}$ *is a left adjoint functor to* $\mathfrak{ct2beh}$ *from Corollary 2 and* $\overline{\tau}^*(r, a, r') = \overline{\tau}(r')$*, and* $\mathfrak{beh2tct}(\lambda) = \lambda$.

Note that $\mathfrak{beh2tct}$ is indeed a functor, because $\mathfrak{G}$ is a functor and $\overline{\tau}(r) \leq^+ \overline{\tau}(r')$ for all runs r and r' such that there is a morphism $m : r \to r'$ in $\mathbf{Beh}(\mathcal{C})$.

Theorem 6. *The functor* $\mathfrak{beh2tct}$ *is a left adjoint to* $\mathfrak{tct2beh}$*. The adjunction is a coreflection, i.e. the unit is a (natural) isomorphism.*

Proof. We show that there is a natural bijection

$$Hom_{\mathbb{TCT}}(\mathfrak{beh2tct}(\mathbf{Beh}(\mathcal{C}), \overline{\tau}), (\mathcal{C}', \tau')) \cong Hom_{\mathbf{TCT}}((\mathbf{Beh}(\mathcal{C}), \overline{\tau}), (\mathbf{Beh}(\mathcal{C}'), \overline{\tau}')).$$

Suppose $f : \mathfrak{beh2tct}(\mathbf{Beh}(\mathcal{C}), \overline{\tau}) \to (\mathcal{C}', \tau')$ is a morphism in $\mathbb{TCT}$. This means that $f : \mathfrak{G}(\mathbf{Beh}(\mathcal{C})) \to \mathcal{C}'$ is a morphism of $\mathbb{CT}$ and $\tau'(f(r), a, f(r')) \leq^+ \overline{\tau}(r')$ for all transitions (r, a, r') in $\mathfrak{G}(\mathbf{Beh}(\mathcal{C}))$. Since $\mathfrak{G}$ is a left adjoint to $\mathfrak{ct2beh}$, there is a morphism $\lambda : \mathbf{Beh}(\mathcal{C}) \to \mathbf{Beh}(\mathcal{C}')$ in $\mathbf{CT}$. Moreover, $\lambda(r) = r_{f(r)}$ for all objects r in $\mathbf{Beh}(\mathcal{C})$. To see that $\lambda : (\mathbf{Beh}(\mathcal{C}), \overline{\tau}) \to (\mathbf{Beh}(\mathcal{C}'), \overline{\tau}')$ is a morphism in $\mathbf{TCT}$, we simply check that λ satisfies the property that $\overline{\tau}' \circ \lambda(r) \leq^+ \overline{\tau}(r)$ for all objects r in $\mathbf{Beh}(\mathcal{C})$. Take an arbitrary run r_n in $\mathcal{C}$. Assume that r_i $(i = 1, \ldots, n)$ is a prefix of r_n such that the length of r_i is equal to i. Then, $\overline{\tau}' \circ \lambda(r_n) = \overline{\tau}'(r_{f(r_n)}) = \bigoplus_{i=1}^{n} \tau'(f(r_{i-1}), a_i, f(r_i)) \leq^+ \bigoplus_{i=1}^{n} \overline{\tau}^*(r_{i-1}, a_i, r_i) = \bigoplus_{i=1}^{n} \overline{\tau}(r_i) = \overline{\tau}(r_n)$.

Conversely, suppose that $\mu : (\mathbf{Beh}(\mathcal{C}), \overline{\tau}) \to (\mathbf{Beh}(\mathcal{C}'), \overline{\tau}')$ is a morphism in **TCT**. Hence, $\mu : \mathbf{Beh}(\mathcal{C}) \to \mathbf{Beh}(\mathcal{C}')$ is a morphism in **CT** and $\overline{\tau}' \circ \mu(r) \leq^+ \overline{\tau}(r)$ for all objects r in $\mathbf{Beh}(\mathcal{C})$. Due to the fact that $\mathfrak{G}$ is a left adjoint to $\mathfrak{ct2beh}$, we can find a morphism $g : \mathfrak{G}(\mathbf{Beh}(\mathcal{C})) \to \mathcal{C}'$ in $\mathbb{CT}$ such that $g(r_n) = last(\mu(r_n))$. We only need to show that $\tau'(g(r), a, g(r')) \leq^+ \overline{\tau}(r')$ for all transitions (r, a, r') in $\mathfrak{G}(\mathbf{Beh}(\mathcal{C}))$. Take an arbitrary run r_n in $\mathbf{Beh}(\mathcal{C})$. Define r_i $(i = 1, \ldots, n)$ as a prefix of r_n such that the length of r_i is equal to i. Note that $\tau'(g(r_{n-1}), a, g(r_n)) = \tau'(last(\mu(r_{n-1})), a, last(\mu(r_{n-1}))) \leq^+ \overline{\tau}'(\mu(r_n)) \leq^+ \overline{\tau}(r_n)$.

These two constructions undoubtedly delineate the needed natural bijection.

By Corollary 2, the unit $\eta : \mathbf{Beh}(\mathcal{C}) \to \mathfrak{ct2beh}(\mathfrak{G}(\mathbf{Beh}(\mathcal{C})))$ is a isomorphism of **CT**, and $\eta(r) = r_r$. Note that $\overline{\overline{\tau^*}} \circ \eta(r) = \overline{\overline{\tau^*}}(r_r) = \bigoplus_{(r,a,r') \in r_r} \overline{\tau}^*(r, a, r')$ $= \bigoplus_{(r,a,r') \in r_r} \overline{\tau}(r') = \bigoplus_{(r,a,r') \in r_r} \bigoplus_{(s,b,s') \in r} \tau(s, b, s') = \bigoplus_{(s,b,s') \in r} \tau(s, b, s') = \overline{\tau}(r)$. Hence, η is a (natural) isomorphism in $\mathbb{TCT}$.

Thus, **TCT** embeds fully and faithfully into $\mathbb{TCT}$ and is equivalent to the full subcategory of $\mathbb{TCT}$ consisting of those objects $(\mathcal{C}, \tau)$ that are isomorphic to the object $\mathfrak{beh2tct}(\mathfrak{tct2beh}(\mathcal{C}, \tau))$. The coreflection mentioned above shows that there exists a morphism from $(\mathcal{C}, \tau)$ to $(\mathcal{C}', \tau')$ in $\mathbb{TCT}$ only when every run of $(\mathcal{C}, \tau)$ with time t can be simulated by a run of $(\mathcal{C}', \tau')$ with time t' such that $t' \leq^+ t$. This gives us a precise correspondence between the morphisms of timed causal trees which consider only their static structures and morphisms on their dynamic behaviors.

6. Conclusion

In this paper, we have established an exact connection between the two distinct ways of the category theory application to the study of causal trees and timed causal trees. The first approach proposes that the models under investigation are represented as structural categories in which the morphisms consider only the static structure of trees. According to the other approach, semantic categories are defined for models; the objects of these categories are behavior categories (which present all possible behaviors of the models) and morphisms consider purely dynamic behavior of models. For causal trees, we have proved that the connection is equivalence. In other words, we have established that the syntactic and semantic categories of causal trees are equivalent. On the other hand, in the case of timed causal trees, this connection is a coreflection between a semantic category whose objects are behavior categories with timings and the structural category of timed causal trees. This means that for timed causal trees the semantic category embeds fully and faithfully into the structural category and is equivalent to some full subcategory of the structural category.

In the future, we plan to extend the obtained results to other classes of models (e.g. timed event structures, networks of timed automata, etc.).

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