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Divergence formulas (conservation laws) for families of curves and surfaces and applications*

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Abstract. The divergence formulas we have obtained (differential conservation laws) of the form $\operatorname{div} \boldsymbol{F} = 0$ for an arbitrary smooth field of unit vectors $\boldsymbol{\tau}(x, y, z)$, for a family of spatial curves as well as $\{L_\tau\}$ for a family of surfaces $\{S_\tau\}$ continuously filling a certain domain. The solenoidal vector field $\boldsymbol{F}$ in these formulas is expressed, respectively, through the field $\boldsymbol{\tau}(x, y, z)$, the characteristics of the curves L_τ and characteristics of the surfaces S_τ. Also, we have found the formulas connecting the surface characteristics and those of the curves orthogonal to them. In the case when the curves L_τ and the surfaces S_τ are vector lines of the vector field $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$ with the direction $\boldsymbol{\tau}$ and the surfaces orthogonal to them, the conservation laws found are equivalent to divergence formulas for the field $\boldsymbol{v}$. With these general geometric formulas the divergent identities (differential conservation laws) for the solutions of the eikonal equation $|\operatorname{grad} \tau|^2 = n^2(x, y, z)$, the Poisson equation $u_{xx} + u_{yy} + u_{zz} = -4\pi\rho(x, y, z)$ and for solutions of Euler's hydrodynamic equations are obtained. In the plane case, these formulas transform to the conservation laws obtained earlier.

This paper is a generalization and development of the published works [1–4].

The vector line L_τ of vector fields corresponding to the solutions of the mathematical physics equations, and to the surfaces S_τ orthogonal to them with the normal $\boldsymbol{\tau}$ continuously fill the domain in question. Therefore, in this paper we study not only the properties of fixed curves and surfaces, but the properties of their families.

In [4], we obtained divergence formulas (conservation laws) for the family $\{L_\tau\}$ of the plane curves L_τ with the tangent unit vector $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$ and the unit normal $\boldsymbol{\nu} = \boldsymbol{\nu}(x, y)$ or for an arbitrary smooth field of the unit vectors $\boldsymbol{\tau}(x, y)$ with the vector lines L_τ. These conservation laws have the form $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0 \Leftrightarrow \operatorname{div} \boldsymbol{S}^* = 0$, where $\boldsymbol{S}(\boldsymbol{\tau}) \stackrel{\text{def}}{=} \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} = \boldsymbol{S}^* \stackrel{\text{def}}{=} \boldsymbol{K}_\tau + \boldsymbol{K}_\nu$, $\boldsymbol{K}_\tau = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} = k\boldsymbol{\nu}$, $\boldsymbol{K}_\nu = (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\nu} = \operatorname{rot} \boldsymbol{\nu} \times \boldsymbol{\nu} = -k_\nu \boldsymbol{\tau}$ are curvature vectors of the curves L_τ with the curvature k and the curves L_ν orthogonal to them with the tangent unit vector $\boldsymbol{\nu}$ and the curvature of k_ν. The symbols $(\boldsymbol{a} \cdot \boldsymbol{b})$ and $\boldsymbol{a} \times \boldsymbol{b}$ denote the scalar and the vector products of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, ∇ is the Hamiltonian operator, $(\boldsymbol{v} \cdot \nabla)\boldsymbol{a}$ is the derivative of the vector $\boldsymbol{a}$ in the direction of the unit vector $\boldsymbol{v}$.

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In this paper, we discuss the three-dimensional case, when we have the field of unit vectors $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$, the family of spatial curves L_τ with the Frenet basis ($\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$) [5–7] ($\boldsymbol{\tau}$ is the unit tangent vector, $\boldsymbol{\nu}$ is a principal normal, $\boldsymbol{\beta}$ is binormal), the first curvature k and the second curvature $\varkappa$, and the family $\{S_\tau\}$ of the surfaces S_τ, orthogonal to the curves L_τ, with the normal $\boldsymbol{\tau}$, the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, the principal curvatures k_1 and k_2, the mean curvature $H \overset{\text{def}}{=} (k_1 + k_2)/2$ and the Gaussian curvature $K \overset{\text{def}}{=} k_1 k_2$ [5–7]. All the values $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, k, $\varkappa$ and $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, H, K are vector and scalar fields in D, which is continuously filled by the curves L_τ and the surface S_τ. In the three-dimensional case the value of $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$ is not identically equal to zero in D. The following geometric reason explains this circumstance. In [8], it was found that the value of $\{-\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})/2\}$ is the Gaussian curvature K of the surface S_τ. The plane case corresponds to the cylindrical surfaces S_τ with the directrices L_ν and generatrices parallel to the axis Oz; their Gaussian curvature $K \equiv 0$ and hence, $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0$ in D. However, in the general case, $K \not\equiv 0$ for a single surface S_τ (except for evolving surfaces [5–7]) and, especially, for the family $\{S_\tau\}$, i.e., in D; hence, $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) \not\equiv 0$ in D.

In this paper, we obtain the divergence formula (differential conservation laws) of the form $\operatorname{div} \boldsymbol{F} = 0$ for an arbitrary smooth field of the unit vectors $\boldsymbol{\tau}(x, y, z)$, for the family of spatial curves $\{L_\tau\}$ and for the family of the surfaces $\{S_\tau\}$ with the normal $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$. The solenoidal vector field $\boldsymbol{F}$ in these formulas is expressed, respectively, through the field $\boldsymbol{\tau}(x, y, z)$, the characteristics $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, k, $\varkappa$ of the curves L_τ and the characteristics $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, K, H of the surfaces S_τ. Some of these formulas contain the field $\boldsymbol{S}^*$ that is the sum of three vectors of curvature of the vector lines of the fields $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ and the field $\boldsymbol{S}_l$ that is the sum of three curvature vectors: vector lines of the normal fields $\boldsymbol{\tau}$ surfaces S_τ and two vector lines of fields of their principle directions $\boldsymbol{l}_1$, $\boldsymbol{l}_2$. In [8], it was found that the field $\boldsymbol{S}(\boldsymbol{\tau})$ is of the sum of three curvature vectors: the vector line L_τ of the field $\boldsymbol{\tau}$ and any two geodesic lines (mutually orthogonal) on the surfaces S_τ, orthogonal to the curves L_τ. In the case when the curves L_τ and the surface S_τ are vector lines of the vector field $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$ with the direction $\boldsymbol{\tau}$ and the surfaces, orthogonal to them, the conservation laws obtained are equivalent to the divergence formulas for the field $\boldsymbol{v}$.

With the help of these general geometric formulas, the differential conservation laws for the solutions of the eikonal $|\operatorname{grad} \tau|^2 = n^2(x, y, z)$, the Poisson equation $\Delta u = -4\pi\rho(x, y, z)$ ($\Delta u = u_{xx} + u_{yy} + u_{zz}$) and three-dimensional solutions of Euler's hydrodynamic equations were obtained. In the plane case, the formulas found transform to conservation laws obtained in [2–4]. The symbols $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ denote the right-hand side system of unit vectors along the axes of the Cartesian coordinate system x, y, z.

Lemma 1 [2]. *For any vector field* $\boldsymbol{v} = \boldsymbol{v}(x, y, z) = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k} = |\boldsymbol{v}|\boldsymbol{\tau}$ *defined in* D, *with components* $v_k(x, y, z) \in C^1(D)$, $k = 1, 2, 3$, *the modulus* $|\boldsymbol{v}| \neq 0$ *in* D *and the direction* $\boldsymbol{\tau} = \boldsymbol{v}/|\boldsymbol{v}|$ $(|\boldsymbol{\tau}| \equiv 1)$ *the following identity holds*:

$$\boldsymbol{T}(\boldsymbol{v}) = \boldsymbol{S}(\boldsymbol{\tau}), \tag{1}$$

where

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{\tau}) &\stackrel{\text{def}}{=} \operatorname{rot}\boldsymbol{\tau} \times \boldsymbol{\tau} - \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} - \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau}, \\ \boldsymbol{T}(\boldsymbol{v}) &\stackrel{\text{def}}{=} \operatorname{grad}\ln|\boldsymbol{v}| + \{\operatorname{rot}\boldsymbol{v} \times \boldsymbol{v} - \boldsymbol{v}\operatorname{div}\boldsymbol{v}\}/|\boldsymbol{v}|^2. \end{aligned} \tag{2}$$

By the direct calculation one can prove

Lemma 2. *Let* $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z) = \cos\alpha_1\,\boldsymbol{i} + \cos\alpha_2\,\boldsymbol{j} + \cos\alpha_3\,\boldsymbol{k}$ *be the vector field of the unit vectors* $(|\boldsymbol{\tau}| \equiv 1)$ *with the domain of definition* D, α_1, α_2, α_3 *are the direction angles between the vector* $\boldsymbol{\tau}$ *and the axes* x, y, z, *respectively, and* $\boldsymbol{\tau}(x, y, z) \in C^1(D)$. *Then the field* $\boldsymbol{S}(\boldsymbol{\tau})$ *of the form of* (2) *can be represented in any of the forms* $\boldsymbol{S}(\boldsymbol{\tau}) = \sum_{j=1}^{3}\operatorname{grad}\cos\alpha_j \times (\boldsymbol{i}_j \times \boldsymbol{\tau}) = \sum_{j=1}^{3}\cos\alpha_j\operatorname{rot}(\boldsymbol{\tau} \times \boldsymbol{i}_j)$, $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{\Phi}_1 - \operatorname{rot}\boldsymbol{\Psi} = \boldsymbol{\Phi}_2 + \operatorname{rot}\boldsymbol{\Psi}$, *where* $\boldsymbol{i}_1 = \boldsymbol{i}$, $\boldsymbol{i}_2 = \boldsymbol{j}$, $\boldsymbol{i}_3 = \boldsymbol{k}$,

$$\begin{aligned} \boldsymbol{\Phi}_1 &\stackrel{\text{def}}{=} 2\big[\cos\alpha_3\operatorname{rot}(\cos\alpha_2\,\boldsymbol{i}) + \cos\alpha_1\operatorname{rot}(\cos\alpha_3\,\boldsymbol{j}) + \cos\alpha_2\operatorname{rot}(\cos\alpha_1\,\boldsymbol{k})\big], \\ \boldsymbol{\Phi}_2 &\stackrel{\text{def}}{=} -2\big[\cos\alpha_2\operatorname{rot}(\cos\alpha_3\,\boldsymbol{i}) + \cos\alpha_3\operatorname{rot}(\cos\alpha_1\,\boldsymbol{j}) + \cos\alpha_1\operatorname{rot}(\cos\alpha_2\,\boldsymbol{k})\big], \\ \boldsymbol{\Psi} &\stackrel{\text{def}}{=} \cos\alpha_2\cos\alpha_3\,\boldsymbol{i} + \cos\alpha_1\cos\alpha_3\,\boldsymbol{j} + \cos\alpha_1\cos\alpha_2\,\boldsymbol{k}. \end{aligned}$$

From Lemmas 1 and 2 follows

Theorem 1. *Under the conditions of Lemma 2 in* D *the following equivalent divergent identities* (*conservation laws*) *for the field* $\boldsymbol{\tau}$ *are valid*: $\operatorname{div}\{\boldsymbol{S}(\boldsymbol{\tau}) - \boldsymbol{\Phi}_i\} = 0$, $i = 1, 2$. *If* $\boldsymbol{\tau}$ *is the direction of the vector field* $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$, *then under the conditions of Lemma 1 in* D *the following equivalent divergent identities for the field* $\boldsymbol{v}$ *hold*: $\operatorname{div}\{\boldsymbol{T}(\boldsymbol{v}) - \boldsymbol{\Phi}_i(\boldsymbol{v})\} = 0$, $i = 1, 2$, *where* $\boldsymbol{\Phi}_i(\boldsymbol{v})$ *is obtained from* $\boldsymbol{\Phi}_i$ *by replacing* $\boldsymbol{\tau}$ *by* $\{\boldsymbol{v}/|\boldsymbol{v}|\}$ *and* $\cos\alpha_j$ *by* $\{v_j/|\boldsymbol{v}|\}$.

Let us obtain divergent formulas (differential conservation laws), which appear to be of a higher order for the field of the unit vectors $\boldsymbol{\tau}$ or for a family of the curves $\{L_\tau\}$, as compared to the identities of Theorem 1. Let $\{L_\tau\}$ be a family of the curves L_τ with the tangent unit vector $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ continuously filling the domain D. Let:

(D) one and only one curve $L_\tau \in \{L_\tau\}$ passes through each point $(x, y, z) \in D$;

(E) at each point (x, y, z) of any curve $L_\tau \in \{L_\tau\}$ there is a (right) Frenet basis $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$, so that three mutually orthogonal vector fields $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ are defined in D;

(F) $\boldsymbol{\tau}(x, y, z) \in C^2(D)$.

By the direct calculations one can prove

Lemma 3. *Let the family $\{L_\tau\}$ of the curves L_τ with the Frenet unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, the first curvature k and the second curvature $\varkappa$ satisfy the conditions* (D)–(F) *in D. Let the field $\boldsymbol{S}^*$ be the sum of the three curvature vectors*:

$$\begin{aligned} \boldsymbol{S}^* &\stackrel{\text{def}}{=} \boldsymbol{K}_\tau + \boldsymbol{K}_\nu + \boldsymbol{K}_\beta = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} + (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\nu} + (\boldsymbol{\beta} \cdot \nabla)\boldsymbol{\beta} \\ &= \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} + \operatorname{rot} \boldsymbol{\nu} \times \boldsymbol{\nu} + \operatorname{rot} \boldsymbol{\beta} \times \boldsymbol{\beta} \\ &= -\{\boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} + \boldsymbol{\nu} \operatorname{div} \boldsymbol{\nu} + \boldsymbol{\beta} \operatorname{div} \boldsymbol{\beta}\} = \{\boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{S}(\boldsymbol{\nu}) + \boldsymbol{S}(\boldsymbol{\beta})\}/2. \end{aligned}$$

Here $\boldsymbol{K}_\tau = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} = k\boldsymbol{\nu}$, $\boldsymbol{K}_\nu = (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\nu} = \operatorname{rot} \boldsymbol{\nu} \times \boldsymbol{\nu}$, $\boldsymbol{K}_\beta = (\boldsymbol{\beta} \cdot \nabla)\boldsymbol{\beta} = \operatorname{rot} \boldsymbol{\beta} \times \boldsymbol{\beta}$ are curvature vectors of the vector lines L_τ, L_ν, L_β of the fields $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, respectively. Then $\boldsymbol{S}^* = \boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau} \times \boldsymbol{R}^*$ in D, where $\boldsymbol{R}^* \stackrel{\text{def}}{=} \varkappa\boldsymbol{\tau} + k\boldsymbol{\beta} + \boldsymbol{\beta} \operatorname{div} \boldsymbol{\nu} - \boldsymbol{\nu} \operatorname{div} \boldsymbol{\beta} = \boldsymbol{\Phi} + \boldsymbol{S}^* \times \boldsymbol{\tau} = \varkappa\boldsymbol{\tau} + (\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\nu})\boldsymbol{\nu} + (\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\beta})\boldsymbol{\beta}$, $\boldsymbol{\Phi} \stackrel{\text{def}}{=} \varkappa\boldsymbol{\tau} + k\boldsymbol{\beta}$. Here Φ is the Darboux vector [5].

Lemma 3 results in

Theorem 2. *Under the conditions of Lemma 3 the divergent identity* (*conservation law for a family of curves $\{L_\tau\}$*) *holds in D*:

$$\operatorname{div}\{\boldsymbol{\tau} \operatorname{div} \boldsymbol{S}^* - \varkappa \operatorname{rot} \boldsymbol{\tau} - k \operatorname{rot} \boldsymbol{\beta}\} = 0 \quad \Leftrightarrow$$
$$\operatorname{div}\Big\{\frac{1}{2}\boldsymbol{\tau} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) - k\boldsymbol{\nu}(\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta}) - k\boldsymbol{\beta}[(\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta}) + \varkappa]\Big\} = 0.$$

Everywhere here the expression in braces is equal to $\operatorname{rot} \boldsymbol{R}^*$; $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 2(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{R}^*)$; $\operatorname{div} \boldsymbol{S}^* = (1/2) \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) + k(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\beta}) + \varkappa(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau})$; the fields $\boldsymbol{S}(\boldsymbol{\tau})$, $\boldsymbol{S}^*$, $\boldsymbol{R}^*$, k, $\varkappa$ are defined in Lemmas 1, 3.

The expressions for the value $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$, the first curvature k and the second curvature $\varkappa$ of the curves L_τ in terms of the fields of the Frenet unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ are given by

Lemma 4. *Under the conditions of Theorem 2, the following identities hold in D*:

$$k = (\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\tau}), \quad \varkappa = \{(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau}) - (\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\nu}) - (\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta})\}/2,$$

$$\begin{aligned}\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) &= 2\{\varkappa[\varkappa - (\boldsymbol{\tau}\cdot \operatorname{rot}\boldsymbol{\tau})] - (\boldsymbol{\tau}\cdot[\operatorname{rot}\boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\beta}])\}\\ &= 2\{(\boldsymbol{\nu}\cdot\operatorname{rot}\boldsymbol{\beta})(\boldsymbol{\beta}\cdot\operatorname{rot}\boldsymbol{\nu}) + [A^2 - (\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau})^2]/4\}\\ A^2 + B^2 &= (\operatorname{div}\boldsymbol{\tau})^2 + 2\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) + (\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau})^2,\end{aligned}$$

where $A \overset{\text{def}}{=} (\boldsymbol{\nu}\cdot\operatorname{rot}\boldsymbol{\nu}) - (\boldsymbol{\beta}\cdot\operatorname{rot}\boldsymbol{\beta})$, $\boldsymbol{B} \overset{\text{def}}{=} (\boldsymbol{\beta}\cdot\operatorname{rot}\boldsymbol{\nu}) + (\boldsymbol{\nu}\cdot\operatorname{rot}\boldsymbol{\beta})$.

Let us find a divergence formula (a conservation law) for the surfaces S_τ given by some general properties in terms of their geometric characteristics. Let $\{S_\tau\}$ be a family of the surfaces S_τ with the unit normal $\boldsymbol{\tau}$ continuously filling the domain D in the space of x, y, z. The principal direction on S_τ will be represented by a unit vector $\boldsymbol{l}_i$ $(i = 1, 2)$ with the corresponding direction; $\boldsymbol{l}_i$ is the unit tangent vector of the curvature lines L_i on S_τ [5–7]. Let:

(A) through each point $(x, y, z) \in D$ there passes one and only one surface $S_\tau \in \{S_\tau\}$;

(B) at each point $(x, y, z) \in D$ there exists a system (the right-hand side one) of the mutually orthogonal unit vectors $\boldsymbol{\tau}$, $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, where $\boldsymbol{\tau}$ is the unit normal, $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are the principal directions on the surface S_τ, passing through this point. For this, it is sufficient that each surface $S_\tau \in \{S_\tau\}$ be C^2-regular [7]. Thus, three mutually orthogonal vector fields of the unit vectors $\boldsymbol{\tau}(x, y, z)$, $\boldsymbol{l}_1(x, y, z)$, $\boldsymbol{l}_2(x, y, z)$ are defined in D. Simultaneously $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ is the tangent unit vector of the curves L_τ, orthogonal to the family $\{S_\tau\}$, that is, the vector lines of the normal vector field $\boldsymbol{\tau}$;

(C) $\boldsymbol{\tau} \in C^n(D)$ (below $n = 1$ or 2), $\boldsymbol{l}_i \in C^1(D)$, $i = 1, 2$.

The geometric meaning of the quantities $\operatorname{div}\boldsymbol{\tau}$, $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$ and the divergent representations of the mean and the Gaussian curvatures of the surfaces is proved in

Theorem 3 [8]. *At any point* $(x, y, z) \in D$ *under the conditions* (A)–(C) *the mean curvature* H *for* $n = 1$ *and the Gaussian curvature* K *for* $n = 2$ *of the surface* S_τ *passing through this point are equal to the divergence* (*the sources density*) *of the vector fields* $\{-\boldsymbol{\tau}/2\}$ *and* $\{-\boldsymbol{S}(\boldsymbol{\tau})/2\}$, *respectively, at this point*: $H = -\frac{1}{2}\operatorname{div}\boldsymbol{\tau}$, $K = -\frac{1}{2}\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$.

The geometric meaning of the conservation law of Theorem 1 explains

Corollary 1. *Under the conditions of Theorem 3, the Gaussian curvature* K *of the surfaces* $S_\tau \in \{S_\tau\}$ *admits in* D *the divergent representation of* $K = -\frac{1}{2}\operatorname{div}\boldsymbol{\Phi}_i$. *If the surfaces* S_τ *are orthogonal to the vector lines* L_τ

of the vector field $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$, $|\boldsymbol{v}| \neq 0$ *in* D *and* $\boldsymbol{v} \in C^2(D)$, *then* $K = -\frac{1}{2}\operatorname{div}\boldsymbol{T}(\boldsymbol{v}) = -\frac{1}{2}\operatorname{div}\boldsymbol{\Phi}(\boldsymbol{v})$. *Here the fields* $\boldsymbol{S}(\boldsymbol{\tau})$, $\boldsymbol{T}(\boldsymbol{v})$, $\boldsymbol{\Phi}_i$, $\boldsymbol{\Phi}(\boldsymbol{v})$ *are defined in Lemmas 1, 2.*

The connection between the characteristics $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, H, K of the surfaces $S_\tau \in \{S_\tau\}$ and the characteristics $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, k, $\varkappa$ of the curves L_τ orthogonal to S_τ is given by

Theorem 4. *Let the family* $\{S_\tau\}$ *of the surfaces* S_τ *with the unit normal* $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ *satisfy the conditions* (A)–(C) *for* $n = 2$ *and the family* $\{L_\tau\}$ *of the curves* L_τ, *orthogonal to* $\{S_\tau\}$, *satisfy the conditions* (D)–(F). *Then at each point* $(x, y, z) \in D$, *the principal directions* $\boldsymbol{l}_1$ *and* $\boldsymbol{l}_2$, *the principal curvatures* k_1 *and* k_2, *the mean curvature* H *and the Gaussian curvature* K *of the surface* S_τ *passing through this point are expressed in terms of the Frenet unit vectors* $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, *the first curvature* k *and the second curvature* $\varkappa$ *of the curves* L_τ *by the formulas*:

$$\begin{gathered}
\boldsymbol{l}_1 = \cos\omega\,\boldsymbol{\nu} + \sin\omega\,\boldsymbol{\beta}, \quad \boldsymbol{l}_2 = -\sin\omega\,\boldsymbol{\nu} + \cos\omega\,\boldsymbol{\beta}, \\
\operatorname{tg} 2\omega = -\frac{A}{B} \quad \Leftrightarrow \quad (\boldsymbol{l}_1 \cdot \operatorname{rot}\boldsymbol{l}_1) = (\boldsymbol{l}_2 \cdot \operatorname{rot}\boldsymbol{l}_2), \\
k_1 = -\frac{1}{2}\{\operatorname{div}\boldsymbol{\tau} \pm \sqrt{A^2 + B^2}\} = -(\boldsymbol{l}_2 \cdot \operatorname{rot}\boldsymbol{l}_1), \\
k_2 = -\frac{1}{2}\{\operatorname{div}\boldsymbol{\tau} \mp \sqrt{A^2 + B^2}\} = (\boldsymbol{l}_1 \cdot \operatorname{rot}\boldsymbol{l}_2), \\
\Rightarrow \quad K \stackrel{\text{def}}{=} k_1 k_2 = \frac{1}{4}\{(\operatorname{div}\boldsymbol{\tau})^2 - (A^2 + B^2)\} \\
K = (\boldsymbol{\tau} \cdot [\operatorname{rot}\boldsymbol{\nu} \times \operatorname{rot}\boldsymbol{\beta}]) - \varkappa^2 = -\Big\{(\boldsymbol{\nu} \cdot \operatorname{rot}\boldsymbol{\beta})(\boldsymbol{\beta} \cdot \operatorname{rot}\boldsymbol{\nu}) + \frac{1}{4}A^2\Big\} \\
\Rightarrow \quad H \stackrel{\text{def}}{=} \frac{1}{2}(k_1 + k_2) = -\frac{1}{2}\operatorname{div}\boldsymbol{\tau}, \qquad K = -\frac{1}{2}\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}),
\end{gathered}$$

where the quantities A, B, k, $\varkappa$, $\boldsymbol{S}(\boldsymbol{\tau})$, $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$ *are given by the formulas of Lemmas 1, 2, 4. Let us place the sign "plus" in front of the radical for* $k_1 < k_2$ *and "minus" for* $k_1 > k_2$. *We have* $H^2 - K = A^2 + B^2 \geq 0 \Rightarrow H^2 \geq K$.

Lemma 5. *Let the conditions of Theorem 4 be valid and the field* $\boldsymbol{S}_l^*$ *be the sum of the three curvature vectors*:

$$\begin{aligned}
\boldsymbol{S}_l^* &\stackrel{\text{def}}{=} \boldsymbol{K}_\tau + \boldsymbol{K}_1 + \boldsymbol{K}_2 = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} + (\boldsymbol{l}_1 \cdot \nabla)\boldsymbol{l}_1 + (\boldsymbol{l}_2 \cdot \nabla)\boldsymbol{l}_2 \\
&= \operatorname{rot}\boldsymbol{\tau} \times \boldsymbol{\tau} + \operatorname{rot}\boldsymbol{l}_1 \times \boldsymbol{l}_1 + \operatorname{rot}\boldsymbol{l}_2 \times \boldsymbol{l}_2 \\
&= -\{\boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{l}_1\operatorname{div}\boldsymbol{l}_1 + \boldsymbol{l}_2\operatorname{div}\boldsymbol{l}_2\} = \{\boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{S}(\boldsymbol{l}_1) + \boldsymbol{S}(\boldsymbol{l}_2)\}/2.
\end{aligned}$$

Here $\boldsymbol{K}_\tau = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} = \operatorname{rot}\boldsymbol{\tau}\times\boldsymbol{\tau} = k\boldsymbol{\nu}$ *is the curvature vector of the vector line* L_τ *of the normal field* $\boldsymbol{\tau}$ *of the surfaces* S_τ, $\boldsymbol{K}_i = (\boldsymbol{l}_i\cdot\nabla)\boldsymbol{l}_i = \operatorname{rot}\boldsymbol{l}_i\times\boldsymbol{l}_i$ *is the curvature vector of the curvature lines* L_i *on* S_τ $(i=1,2)$*. Then in* D

$$\begin{aligned}\boldsymbol{S}_l^* = \boldsymbol{S}^* + \boldsymbol{\tau}\times\operatorname{grad} w, \quad \boldsymbol{S}_l^* &= \boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau}\times\boldsymbol{R}_l^* \quad \Rightarrow\\ \boldsymbol{R}_l^* \overset{\text{def}}{=} \operatorname{grad} w + \boldsymbol{R}^* &= \varkappa_l\boldsymbol{\tau} + k\boldsymbol{\beta} + \boldsymbol{S}_l^*\times\boldsymbol{\tau}\\ &= \varkappa_l\boldsymbol{\tau} + \operatorname{rot}\boldsymbol{\tau} - (\boldsymbol{l}_1\operatorname{div}\boldsymbol{l}_2 - \boldsymbol{l}_2\operatorname{div}\boldsymbol{l}_1)\\ &= \varkappa_l\boldsymbol{\tau} + \boldsymbol{l}_1(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{l}_1) + \boldsymbol{l}_2(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{l}_2),\end{aligned}$$

where $\boldsymbol{S}^*$, $\boldsymbol{R}^*$, w *are defined in Theorems 2, 4, and*

$$\varkappa_l \overset{\text{def}}{=} -\{(\boldsymbol{l}_1\cdot\operatorname{rot}\boldsymbol{l}_1) + (\boldsymbol{l}_2\cdot\operatorname{rot}\boldsymbol{l}_2)\}/2 = -(\boldsymbol{l}_i\cdot\operatorname{rot}\boldsymbol{l}_i), \quad i=1,2.$$

The expression of the conservation law of Theorem 2 in terms of the characteristics of the surfaces S_τ is given by

Theorem 5. *With the conditions and notations of Lemma 5 and Theorem 4 for the family* $\{S_\tau\}$ *of the surfaces* S_τ *in* D *there holds the divergent identity (conservation law):*

$$\begin{aligned}&\operatorname{div}\{K\boldsymbol{\tau} + k_2(\boldsymbol{l}_2\cdot\operatorname{rot}\boldsymbol{\tau})\boldsymbol{l}_1 - k_1(\boldsymbol{l}_1\cdot\operatorname{rot}\boldsymbol{\tau})\boldsymbol{l}_2\} = 0\\ \Leftrightarrow\quad &\operatorname{div}\{K\boldsymbol{\tau} + (H+B/2)\boldsymbol{K}_\tau - A\operatorname{rot}\boldsymbol{\tau}/2\} = 0\\ \Leftrightarrow\quad &\operatorname{div}\{-\boldsymbol{\tau}\operatorname{div}\boldsymbol{S}_l^* + (\boldsymbol{l}_1\cdot\operatorname{rot}\boldsymbol{\tau})\operatorname{rot}\boldsymbol{l}_1 + (\boldsymbol{l}_2\cdot\operatorname{rot}\boldsymbol{\tau})\operatorname{rot}\boldsymbol{l}_2 + \varkappa_l\operatorname{rot}\boldsymbol{\tau}\}.\end{aligned}$$

Here, the expression in braces { } *in everywhere equal to* $\{-\operatorname{rot}\boldsymbol{R}_l^*\} = -\operatorname{rot}\boldsymbol{R}^*$*;* $K = -(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{R}_l^*)$*;* $\operatorname{div}\boldsymbol{S}_l^* = -K + (\boldsymbol{l}_1\cdot\operatorname{rot}\boldsymbol{\tau})(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{l}_1) + (\boldsymbol{l}_2\cdot\operatorname{rot}\boldsymbol{\tau})(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{l}_2)$*.*

Remark 1. The Frenet unit vectors $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ and the first curvature k of the curves L_τ can be expressed in terms of $\boldsymbol{\tau}$: $\boldsymbol{\nu} = (\operatorname{rot}\boldsymbol{\tau}\times\boldsymbol{\tau})/k$, $\boldsymbol{\beta} = \boldsymbol{\tau}\times\boldsymbol{\nu}$, $k = |\operatorname{rot}\boldsymbol{\tau}\times\boldsymbol{\tau}|$. Since by the formulas of Lemma 4 and Theorem 4 the quantities $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, $\varkappa$, $\boldsymbol{S}(\boldsymbol{\tau})$, H, K are expressed in terms of the orts $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ of the curves L_τ, then finally, all these quantities can be expressed only through the field $\boldsymbol{\tau}$. Therefore, all the formulas of Theorems 2, 4, 5 and Lemmas 3–5 can be only expressed through the field $\boldsymbol{\tau}$ (the unit tangent vectors of the curves L_τ or normals to S_τ).

Let us apply the general formulas obtained to the solutions of the mathematical physics equations. Theorem 1 implies

Corollary 2. *Let* $\tau = \tau(x,y,z)$ *be the solution of the eikonal equation* $\tau_x^2 + \tau_y^2 + \tau_z^2 = n^2(x,y,z)$ *in* D*, the time field* $\tau\in C^3(D)$*, the refractive index* $n\in C^2(D)$*. Then in* D *the following conservation law holds*:

$$\operatorname{div}\{\boldsymbol{T} - \boldsymbol{\Phi}_i(\tau)\} = 0, \qquad i = 1, 2,$$

where $\boldsymbol{T} = \operatorname{grad}\ln n - \Delta\tau \operatorname{grad}\tau/n^2$, $\boldsymbol{\Phi}_i(\tau)$ *are obtained from the values* $\boldsymbol{\Phi}_i$ *of Lemma 2 by substitution* $\cos\alpha_1 = \tau_x/n$, $\cos\alpha_2 = \tau_y/n$, $\cos\alpha_3 = \tau_z/n$.

Corollary 3. *Let* $u = u(x, y, z)$ *be the solution of the Poisson equation* $\Delta u = -4\pi\rho(x, y, z)$ *in* D, $|\operatorname{grad} u| \neq 0$ *in* D, *the potential* $u \in C^3(D)$, *the density* $\rho \in C^1(D)$. *Then, in* D *the conservation law* $\operatorname{div}\{\boldsymbol{T} - \boldsymbol{\Phi}_i(u)\} = 0$ *holds, where* $\boldsymbol{T} = \operatorname{grad}\ln|\operatorname{grad} u| + 4\pi\rho \operatorname{grad} u/|\operatorname{grad} u|^2$, *the fields* $\boldsymbol{\Phi}_i(u)$ *are defined in Corollary 2 by replacing* τ *by* u.

Corollary 4. *Let* $\boldsymbol{v} = \boldsymbol{v}(z, y, z) = v\boldsymbol{\tau}$ *be the velocity in Euler's hydrodynamic equations* $\boldsymbol{v}_t + \operatorname{grad} v^2/2 - \boldsymbol{v} \times \operatorname{rot}\boldsymbol{v} = \boldsymbol{F} - \operatorname{grad} p/\rho$, *which can be written down in* D *as* $\boldsymbol{G} = -\boldsymbol{T}(\boldsymbol{v})$ $(= -\boldsymbol{S}(\boldsymbol{\tau}))$, *where* $\boldsymbol{G} \overset{\text{def}}{=} \{\boldsymbol{v}_t + \boldsymbol{v}\operatorname{div}\boldsymbol{v} + \operatorname{grad} p/\rho - \boldsymbol{F}\}/v^2$; $v \overset{\text{def}}{=} |\boldsymbol{v}| \neq 0$ *in* D, $\boldsymbol{v} \in C^2(D)$, *the pressure* $p \in C^2(D)$, *the density* $\rho \in C^1(D)$, *the body force per unit of mass* $\boldsymbol{F} \in C^1(D)$. *Then in* D *the conservation law* $\operatorname{div}\{\boldsymbol{G} + \boldsymbol{\Phi}_i(\boldsymbol{v})\} = 0$, $i = 1, 2$, *holds, where the field* $\boldsymbol{\Phi}_i(\boldsymbol{v})$ *is defined in Theorem 1.*

Similarly by virtue of Theorems 2, 5, Remark 1, and the equality $\boldsymbol{\tau} = \boldsymbol{v}/|\boldsymbol{v}|$, we obtain for the above equations for the conservations laws of a higher order. In this case, the role of mutually orthogonal families of the curves L_τ and the surfaces S_τ for the eikonal equation is played by rays (the vector lines of the field $\boldsymbol{v} = \operatorname{grad}\tau$) and by the fronts $\tau(x, y, z) = \text{const}$; for the Poisson equation — by the vector (force) lines of the field $\boldsymbol{v} = \operatorname{grad} u$ and by the equipotential surfaces $u(x, y, z) = \text{const}$; for Euler's hydrodynamic equations — by the streamlines (the vector lines of the velocity field $\boldsymbol{v}$ at $t = \text{const}$) and by surfaces orthogonal to them. In the plane case, when all the quantities are independent of z, from Theorems 1, 2, 5 and from the Corollaries 2, 4 follow the conservation laws $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = \operatorname{div}\boldsymbol{S}^* = 0$, $\operatorname{div}\boldsymbol{T} = 0$, and $\operatorname{div}\boldsymbol{G} = 0$ obtained in [2–4].

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