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Some formulas for families of curves and surfaces and their applications

A.G. Megrabov

Abstract. A unit vector field $\boldsymbol{\tau}$ in the Euclidean space E^3 is considered. Let $\boldsymbol{P}$ be the vector field from the first Aminov divergent representation $K = \operatorname{div}[(\boldsymbol{r}\cdot\boldsymbol{\tau})\boldsymbol{P}]$ for the total curvature of the second kind K of the field $\boldsymbol{\tau}$. For the field $\boldsymbol{P}$, an invariant representation of the form $\boldsymbol{P} = -\operatorname{rot}\boldsymbol{R}^*$ is obtained, where the field $\boldsymbol{R}^*$ is expressed in terms of the Frenet basis $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$ and the first curvature k and the second curvature $\varkappa$ of the streamlines L_τ of the field $\boldsymbol{\tau}$. Formulas relating to the quantities K (or $\boldsymbol{P}$), $\varkappa$, $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ are derived.

Three-dimensional analogs to the conservation law $\operatorname{div}\boldsymbol{S}_p^* = 0$, which is valid for a family of plane curves L_τ, are obtained, where $\boldsymbol{S}_p^*$ is the sum of the curvature vectors of the plane curves L_τ and their orthogonal curves L_ν. It is shown that if the field $\boldsymbol{\tau}$ is holonomic: 1) the vector field $\boldsymbol{S}(\boldsymbol{\tau})$ from the second Aminov divergent representation $K = -\frac{1}{2}\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$ can be interpreted as the sum of three curvature vectors of three curves related to surfaces S_τ with the normal $\boldsymbol{\tau}$; 2) the non-holonomicity values of the fields of the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are equal. Applications of the obtained geometric formulas to the equations of mathematical physics are discussed.

Keywords: vector field, total curvature, family of curves, family of surfaces, conservation laws.

1. Introduction

1.1. The vector physical fields described by the equations of mathematical physics have vector lines L_τ (e.g., the rays for the eikonal equation or the streamlines for the Euler hydrodynamic equations) which form a family of curves $\{L_\tau\}$ and continuously fill a domain D in the three-dimensional space. The surfaces S_τ with the normal $\boldsymbol{\tau}$ which are orthogonal to the curves L_τ (if such surfaces S_τ exist), e.g., wavefronts for the eikonal equation, also form a family $\{S_\tau\}$. It is therefore of interest to study not only the properties of a fixed curve L_τ or a fixed surface S_τ but also the properties of a family of curves $\{L_\tau\}$ or a family of surfaces $\{S_\tau\}$ which continuously fill a domain D.

In this paper, we consider the three-dimensional case where we have a unit vector field $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$, a family of spatial curves L_τ with the Frenet basis $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$ [1] ($\boldsymbol{\tau}$ is the unit tangent vector, $\boldsymbol{\nu}$ is the unit principal normal vector, $\boldsymbol{\beta}$ is the unit binormal vector), the first curvature k and the second curvature $\varkappa$, and the family $\{S_\tau\}$ of the surfaces S_τ which are orthogonal to the curves L_τ and have the normal $\boldsymbol{\tau}$, the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, the principal curvatures k_1 and k_2, the mean curvature $H \stackrel{\text{def}}{=} (k_1 + k_2)/2$ and the Gaisusian curvature $K \stackrel{\text{def}}{=} k_1 k_2$ [1]. All the quantities $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, k, $\varkappa$, and $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, H, K are the vector and the scalar fields in the domain D.

1.2. Assume that D is a domain in the Euclidean space E^3 with the Cartesian coordinates x, y, z; $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the unit vectors along the coordinate axes x, y, and z, respectively; $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z) = \tau_1 \boldsymbol{i} + \tau_2 \boldsymbol{j} + \tau_3 \boldsymbol{k}$ is the unit vector field defined in D, and $\tau_k = \tau_k(x, y, z)$ are the scalar functions $(k = 1, 2, 3)$, $|\boldsymbol{\tau}|^2 = 1$. The geometry of vector fields (see [2]) considers the case of a holonomic field $\boldsymbol{\tau}$ for which there is a family of surfaces S_τ with the normal $\boldsymbol{\tau}$ which are orthogonal to the field $\boldsymbol{\tau}$ and the general case, where the field $\boldsymbol{\tau}$ can be non-holonomic. A necessary and sufficient condition for the holonomicity of the field $\boldsymbol{\tau}$ [2, Ch.1,§ 1] is the fulfillment of the identity $\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$ in D. The geometry of vector fields introduces analogs to the classical characteristics of the surfaces S_τ for a non-holonomic field $\boldsymbol{\tau}$ [2]. For example, the analog to the Gaussian curvature of the surface S_τ is the total curvature of the second kind K [2]. In the case of a holonomic field $\boldsymbol{\tau}$, these analogs coincide with the corresponding classical characteristics of the surfaces S_τ with the normal $\boldsymbol{\tau}$; for example, the above-mentioned quantity K coincides with the Gaussian curvature [2]. For the quantity K, Yu.A. Aminov (see [2, Ch. 1, § 7; 3]) has obtained the first divergent representation:

$$K = \operatorname{div}[(\boldsymbol{r} \cdot \boldsymbol{\tau})\boldsymbol{P}], \tag{1}$$

where $\boldsymbol{r}$ is the radius vector of the point (x, y, z), and the vector $\boldsymbol{P}$ called the curvature vector of the field $\boldsymbol{\tau}$ has the invariant form [2, Ch. 1, § 10]:

$$\boldsymbol{P} = K\boldsymbol{\tau} - 2 \operatorname{div} \boldsymbol{\tau} \boldsymbol{K}_\tau + (\boldsymbol{K}_\tau \cdot \nabla)\boldsymbol{\tau}, \tag{2}$$

where $\boldsymbol{K}_\tau = k\boldsymbol{\nu} = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = \dfrac{d\boldsymbol{\tau}}{ds} = \boldsymbol{\tau}_s$ is the curvature vector of the curve L_τ with the unit tangent vector $\boldsymbol{\tau}$ and the principal normal $\boldsymbol{\nu}$, L_τ is a streamline or a vector line of the field $\boldsymbol{\tau}$, k is its curvature, $(\boldsymbol{v} \cdot \nabla)\boldsymbol{a}$ is the derivative of the vector $\boldsymbol{a}$ in the direction of the vector $\boldsymbol{v}$, d/ds is the differentiation operator in the direction $\boldsymbol{\tau}$ along the curve L_τ with respect to the natural parameter s; $d\varphi/ds = \varphi_s = \operatorname{grad} \varphi \cdot \boldsymbol{\tau}$ for the scalar function $\varphi(x, y, z)$. The symbols $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ denote the scalar and the vector products of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, ∇ is the Hamiltonian operator (nabla).

1.3. Assume that $\{L_\tau\}$ is a family of curves L_τ which continuously fill the domain D and:

(A) one and only one curve $L_\tau \in \{L_\tau\}$ passes through each point $(x, y, z) \in D$;

(B) at each point (x, y, z) of any curve $L_\tau \in \{L_\tau\}$ there is a right-hand Frenet basis $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$ ($\boldsymbol{\beta}$ is the binormal), so that the three mutually orthogonal vector fields $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ are defined in D; $\boldsymbol{\tau} = \boldsymbol{\nu} \times \boldsymbol{\beta}$, $\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}$, and $\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$;

(C) $\boldsymbol{\tau}(x, y, z) \in C^2(D)$.

It this paper (Section 2, Theorem 3), we will show that under conditions (A)–(C), the field $\boldsymbol{P}$ of the form (2) from formula (1) can be represented as

$$\boldsymbol{P} = -\operatorname{rot} \boldsymbol{R}^*, \tag{3}$$

where the vector field $\boldsymbol{R}^*$ can be given by any of the following invariant representations:

$$\boldsymbol{R}^* \stackrel{\text{def}}{=} \varkappa \boldsymbol{\tau} + k\boldsymbol{\beta} + \boldsymbol{\beta} \operatorname{div} \boldsymbol{\nu} - \boldsymbol{\nu} \operatorname{div} \boldsymbol{\beta}, \tag{4}$$

$$\boldsymbol{R}^* = (\varkappa - \boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau})\boldsymbol{\tau} + \nabla(\boldsymbol{\nu}, \boldsymbol{\beta}) = \boldsymbol{\Phi} + \boldsymbol{S}^* \times \boldsymbol{\tau}, \tag{5}$$

$$\boldsymbol{R}^* = \varkappa \boldsymbol{\tau} + (\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\nu})\boldsymbol{\nu} + (\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\beta})\boldsymbol{\beta}. \tag{6}$$

Here $\varkappa$ is the second curvature of the curve L_τ, $\boldsymbol{\Phi} \stackrel{\text{def}}{=} \varkappa \boldsymbol{\tau} + k\boldsymbol{\beta}$ is the Darboux vector [1], $\nabla(\boldsymbol{\nu}, \boldsymbol{\beta}) \stackrel{\text{def}}{=} (\boldsymbol{\beta} \cdot \nabla)\boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\beta}$ is the Poisson bracket [2] for $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$, $\boldsymbol{S}^*$ is the sum of three curvature vectors of vector lines L_τ, L_ν, and L_β of the fields $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$, respectively. The formulas for the quantities K (or $\boldsymbol{P}$), $\varkappa$, $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ will be derived in Section 2.3.

1.4. Let us introduce the vector field

$$\boldsymbol{S}(\boldsymbol{\tau}) \stackrel{\text{def}}{=} \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} = \boldsymbol{K}_\tau - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau}. \tag{7}$$

In the plane case ($\boldsymbol{\tau} = \boldsymbol{\tau}(x, y) = \tau_1 \boldsymbol{i} + \tau_2 \boldsymbol{j}$, $\tau_3 \equiv 0$, $\theta \equiv \pi/2$, $\boldsymbol{\beta} = \boldsymbol{k}$, $\varkappa = 0$), as shown in [4], we have $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{S}_p^*$, where $\boldsymbol{S}_p^* = \boldsymbol{K}_\tau + \boldsymbol{K}_\nu$ is the sum of the curvature vectors $\boldsymbol{K}_\tau = k\boldsymbol{\nu}$ and $\boldsymbol{K}_\nu = k_\nu \boldsymbol{\eta} = -k_\nu \boldsymbol{\tau}$ of the two plane curves L_τ and L_ν from the mutually orthogonal families $\{L_\tau\}$, $\{L_\nu\}$ (k, $\boldsymbol{\tau}$, and $\boldsymbol{\nu}$ are the curvature, the unit tangent vector, and the unit normal of the curve L_τ, and k_ν, $\boldsymbol{\nu}$, $\boldsymbol{\eta} = -\boldsymbol{\tau}$ are the same quantities for the curve L_v). It has been found [4] that $\operatorname{div} \boldsymbol{S}_p^* = \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0$, i.e., $\boldsymbol{S}_p^*$ is a solenoidal field and $\boldsymbol{S}_p^* = -\operatorname{rot}[\alpha(x, y)\boldsymbol{k}]$, where $\alpha = \alpha(x, y)$ is the angle that the vector $\boldsymbol{\tau}$ makes with the axis Ox: $\boldsymbol{\tau} = \boldsymbol{\tau}(\alpha) = \cos\alpha\, \boldsymbol{i} + \sin\alpha\, \boldsymbol{j}$. The identity $\operatorname{div} \boldsymbol{S}_p^* = 0 \Leftrightarrow \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0$ can be regarded as the law of conservation for the family $\{L_\tau\}$ of plane curves [4]. It explains the geometric meaning of the differential conservation laws for the eikonal equation (here S_p^* is the sum of the curvature vectors of the rays and fronts) and for the Euler's hydrodynamic equations (here $\boldsymbol{S}_p^*$ is the sum of the curvature vectors of streamlines and the curves orthogonal to them) in the two-dimensional case obtained in [5, 6].

As stated in [5], any vector field $\boldsymbol{v} = \boldsymbol{v}(x, y, z) = |\boldsymbol{v}|\boldsymbol{\tau}$ with the direction $\boldsymbol{\tau}$ ($|\boldsymbol{\tau}| \equiv 1$) and modulus $|\boldsymbol{v}| \neq 0$ in D ($\boldsymbol{v} \in C^1(D)$) satisfies the identity $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{T}(\boldsymbol{v})$, where $\boldsymbol{T}(\boldsymbol{v}) = \operatorname{grad} \ln |\boldsymbol{v}| + (\operatorname{rot} \boldsymbol{v} \times \boldsymbol{v} - \boldsymbol{v} \operatorname{div} \boldsymbol{v})/|\boldsymbol{v}|^2$. Therefore, in the plane case, the identity $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0$ is equivalent to the identity $\operatorname{div} \boldsymbol{T}(\boldsymbol{v}) = 0$. In the case $\boldsymbol{v} = \operatorname{grad} u(x, y)$, the latter was obtained (see the references in [4–6]) as vector representation of the formula relating to the differential invariants of the Lie group G. (The group G is an equivalence group of the eikonal equation $u_x^2 + u_y^2 = n^2(x, y)$ and other equations of mathematical physics, as well as an extension of the group of conformal transformations of the plane x, y to the space x, y, t, $u^1 = u$, $u^2 = n^2$.) This formula expresses the Gaussian curvature $K = -\Delta \ln n^2/(2n^2)$ of the surface with the linear element $ds^2 = n^2(x, y)(dx^2 + dy^2)$ in terms of the other differential invariants of the group G. The search for the three-dimensional analogs to the conservation law $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0$ for the plane case, the geometric meaning of the field $\boldsymbol{S}(\boldsymbol{\tau})$, and their applications in mathematical physics has led to the results described in [4–6] and in the present paper.

In the three-dimensional case, the analog to the field $\boldsymbol{S}_p^*$ is naturally defined as the sum $\boldsymbol{S}^* = \boldsymbol{K}_\tau + \boldsymbol{K}_\nu + \boldsymbol{K}_\beta$ of the three curvature vectors of the vector lines of the Frenet unit vector fields $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ of the curves L_τ, and $\boldsymbol{S}(\boldsymbol{\tau}) \neq \boldsymbol{S}^*$. The relationship between the fields $\boldsymbol{S}(\boldsymbol{\tau})$ and $\boldsymbol{S}^*$ is given in Lemma 3; the measure of a difference between $\boldsymbol{S}(\boldsymbol{\tau})$ and $\boldsymbol{S}^*$ is in a sense the field $\boldsymbol{R}^*$. Generally, in the three-dimensional case, $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) \neq 0$ and $\operatorname{div} \boldsymbol{S}^*(\boldsymbol{\tau}) \neq 0$. The three-dimensional scalar and vector analogs to the conservation law $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0 \Leftrightarrow \operatorname{div} \boldsymbol{S}_p^* = 0$ for the plane case is obtained in Section 2.3. Note that the vector field $\boldsymbol{S}(\boldsymbol{\tau})$ enters the second Aminov divergent representation [2, Ch. 1, § 8] for the total curvature K of the second kind of the vector field $\boldsymbol{\tau}$: $K = -\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})/2$ (in this case, $-\operatorname{div} \boldsymbol{\tau} = 2H$, where H is the mean curvature).

1.5. In Sections 3.2 and 3.3, it is shown that in the case of a holonomic field $\boldsymbol{\tau}$:

- the vector field $\boldsymbol{S}(\boldsymbol{\tau})$, as well as the field $\boldsymbol{S}^*$, can be geometrically interpreted as the sum of three curvature vectors of three curves (related to the surfaces S_τ with the normal $\boldsymbol{\tau}$);
- the non-holonomicity values [2, Ch. 1, § 1] the principal direction fields on S_τ are equal.

1.6. Section 4 contains applications of the geometric formulas obtained in Sections 2 and 3 for the equations of mathematical physics.

2. Representation of the field $\boldsymbol{P}$ in the form of $\boldsymbol{P} = -\operatorname{rot} \boldsymbol{R}^*$

2.1. The vector fields $\boldsymbol{S}(\boldsymbol{\tau})$, $\boldsymbol{S}^*$, and $\boldsymbol{R}^*$. We represent the field $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\alpha, \theta) \stackrel{\text{def}}{=} \cos\alpha \sin\theta\, \boldsymbol{i} + \sin\alpha \sin\theta\, \boldsymbol{j} + \cos\theta\, \boldsymbol{k}, \tag{8}$$

where $\alpha = \alpha(x, y, z)$ is the angle that the vector $(\tau_1 \boldsymbol{i} + \tau_2 \boldsymbol{j})$ makes with the axis Ox, so that $\cos\alpha = \tau_1/\sqrt{g}$, $\sin\alpha = \tau_2/\sqrt{g}$, where $g = \tau_1^2 + \tau_2^2$, i.e., $\alpha(x, y, z)$ is the polar angle of the point $(\xi = \tau_1,\ \eta = \tau_2)$ in the plane ξ, η: $\alpha \stackrel{\text{def}}{=} \operatorname{arctg}(\tau_2/\tau_1) + (2k + \delta)\pi$, $k \in \mathbb{Z}$, $\delta = 0$, and $\delta = 1$, respectively, in quadrants I, IV and II, III of the plane ξ, η; $\theta = \theta(x, y, z)$ is the angle between the vector $\boldsymbol{\tau}$ and the axis Oz: $\theta \stackrel{\text{def}}{=} \arccos(\tau_3/|\boldsymbol{v}|)$, so that $0 \le \theta \le \pi$, $\cos\theta = \tau_3$, and $\sin\theta = \sqrt{g}$. This means that α and θ are spherical coordinates in the space $\xi = \tau_1$, $\eta = \tau_2$, $\zeta = \tau_3$.

Lemma 1. *Let conditions* (A)–(C) *be satisfied. Then the field* $\boldsymbol{S}(\boldsymbol{\tau})$ *of the form* (7) *can be represented in* D *as* $\boldsymbol{S}(\boldsymbol{\tau}) = \sin\theta \operatorname{grad}\alpha \times \boldsymbol{\nu}_1 - \operatorname{grad}\theta \times \boldsymbol{\nu}_2$, $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 2(\boldsymbol{\tau} \cdot \sin\theta \boldsymbol{A})$, *where* $\sin\theta\, \boldsymbol{A} = -\operatorname{grad}\alpha \times \operatorname{grad}\cos\theta = \operatorname{rot}(\cos\theta \operatorname{grad}\alpha) = -\operatorname{rot}(\alpha \operatorname{grad}\cos\theta)$, $\boldsymbol{A} \stackrel{\text{def}}{=} \operatorname{grad}\alpha \times \operatorname{grad}\theta$; *the principal normal* $\boldsymbol{\nu}$ *and the binormal* $\boldsymbol{\beta}$ *of the curve* $L_\tau \in \{L_\tau\}$, *and the field* $\boldsymbol{S}(\boldsymbol{\tau})$ *can be represented as* $(k \neq 0)$ $\boldsymbol{\nu} = (\alpha_s \sin\theta\, \boldsymbol{\nu}_2 + \theta_s \boldsymbol{\nu}_1)/k$, $\boldsymbol{\beta} = (-\alpha_s \sin\theta\, \boldsymbol{\nu}_1 + \theta_s \boldsymbol{\nu}_2)/k$, *where* $\boldsymbol{\nu}_1 \stackrel{\text{def}}{=} \cos\alpha\cos\theta\, \boldsymbol{i} + \sin\alpha\cos\theta\, \boldsymbol{j} - \sin\theta\, \boldsymbol{k}$ $(\sin\theta\, \boldsymbol{\nu}_1 = \cos\theta\, \boldsymbol{\tau} - \boldsymbol{k})$, $\boldsymbol{\nu}_2 = -\sin\alpha\, \boldsymbol{i} + \cos\alpha\, \boldsymbol{j}$, $\alpha_s = d\alpha/ds = \operatorname{grad}\alpha \cdot \boldsymbol{\tau}$, $\theta_s = d\theta/ds = \operatorname{grad}\theta \cdot \boldsymbol{\tau}$; $\boldsymbol{S}(\boldsymbol{\tau}) = (\boldsymbol{A}_1 \times \boldsymbol{\nu} - \boldsymbol{A}_2 \times \boldsymbol{\beta})/k$, *where* $\boldsymbol{A}_1 \stackrel{\text{def}}{=} \sin\theta\,(\theta_s \operatorname{grad}\alpha - \alpha_s \operatorname{grad}\theta)$, $\boldsymbol{A}_2 \stackrel{\text{def}}{=} \alpha_s \sin^2\theta \operatorname{grad}\alpha + \theta_s \operatorname{grad}\theta$. *The unit vectors* $(\boldsymbol{\tau}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ *form the right-hand system, i.e.,* $\boldsymbol{\nu}_1 \times \boldsymbol{\nu}_2 = \boldsymbol{\tau}$, $\boldsymbol{\tau} \times \boldsymbol{\nu}_1 = \boldsymbol{\nu}_2$, $\boldsymbol{\nu}_2 \times \boldsymbol{\tau} = \boldsymbol{\nu}_1$.

Proof. From formula (8) we have $\boldsymbol{\tau}_s = \alpha_s \sin\theta\, \boldsymbol{\nu}_2 + \theta_s \boldsymbol{\nu}_1$ and $\operatorname{div}\boldsymbol{\tau} = \sin\theta\,(\operatorname{grad}\alpha \cdot \boldsymbol{\nu}_2) + \operatorname{grad}\theta \cdot \boldsymbol{\nu}_1$, whence using the well-known formulas [7] $\boldsymbol{\nu} = \boldsymbol{\tau}_s/k$ and $\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$ and expressing $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ in terms of $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$, we obtain the formulas of the lemma for $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, and $\boldsymbol{S}(\boldsymbol{\tau})$. The formula for $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$ can be obtained, for example, by rewriting $\boldsymbol{S}(\boldsymbol{\tau})$ in the form $\boldsymbol{S}(\boldsymbol{\tau}) = -\sin^2\theta \operatorname{rot}(\alpha \boldsymbol{k}) - \sin\theta\cos\theta \operatorname{rot}\boldsymbol{\nu}_2 - \cos\alpha \operatorname{rot}(\theta \boldsymbol{j}) + \sin\alpha \operatorname{rot}(\theta \boldsymbol{i})$ and using the well-known formula [7] $\operatorname{div}(\varphi \operatorname{rot} \boldsymbol{a}) = \operatorname{grad}\varphi \cdot \operatorname{rot}\boldsymbol{a}$. □

Lemma 2. *Let conditions* (A)–(C) *be satisfied. Then for the first curvature* k *and the second curvature* $\varkappa$ *of the curve* $L_\tau \in \{L_\tau\}$ *in the domain* D, *the following formulas hold* $(k \neq 0)$: $k^2 = \alpha_s^2 \sin^2\theta + \theta_s^2$, $\varkappa = \varphi_s + \alpha_s \cos\theta$, *and* $\varphi_s = \operatorname{grad}\varphi \cdot \boldsymbol{\tau} = [(\theta_s \alpha_{ss} - \alpha_s \theta_{ss})\sin\theta + \alpha_s \theta_s^2 \cos\theta]/k^2$, *where* $\varphi \stackrel{\text{def}}{=} \operatorname{arctg} \dfrac{\alpha_s \sin\theta}{\theta_s}$, $\alpha_{ss} = \dfrac{d^2\alpha}{ds^2} = \operatorname{grad}\alpha_s \cdot \boldsymbol{\tau}$, *and* $\theta_{ss} = \dfrac{d^2\theta}{ds^2} = \operatorname{grad}\theta_s \cdot \boldsymbol{\tau}$.

Proof. The lemma follows from the well-known formulas [7] $k^2 = |\boldsymbol{\tau}_s|^2$, $\varkappa = ([\boldsymbol{\tau} \times \boldsymbol{\tau}_s] \cdot \boldsymbol{\tau}_{ss})/k^2 = (\boldsymbol{\tau}_{ss} \cdot k\boldsymbol{\beta})/k^2$ (we have $k\boldsymbol{\beta} = \boldsymbol{\tau} \times k\boldsymbol{\nu} = \boldsymbol{\tau} \times \boldsymbol{\tau}_s$), and $\boldsymbol{\tau}_{ss} = (\boldsymbol{\tau}_s)_s$, the expression $\boldsymbol{\tau}_s = \alpha_s \sin\theta\, \boldsymbol{\nu}_2 + \theta_s \boldsymbol{\nu}_1$, and the formulas of Lemma 1 for $\boldsymbol{\beta}$, $\boldsymbol{\nu}_1$, and $\boldsymbol{\nu}_2$ using simple calculations. □

The field $\boldsymbol{R}^*$ included in formula (3) appears in the following

Lemma 3. *Let the family $\{L_\tau\}$ of the curves L_τ with the Frenet unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$, the first curvature k, and the second curvature $\varkappa$ in the domain D satisfy conditions* (A)–(C)*. Let the field $\boldsymbol{S}^*$ be the sum of the three curvature vectors*:

$$\begin{aligned}\boldsymbol{S}^* &\stackrel{\text{def}}{=} \boldsymbol{K}_\tau + \boldsymbol{K}_\nu + \boldsymbol{K}_\beta = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} + (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\nu} + (\boldsymbol{\beta} \cdot \nabla)\boldsymbol{\beta} \\ &= \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} + \operatorname{rot} \boldsymbol{\nu} \times \boldsymbol{\nu} + \operatorname{rot} \boldsymbol{\beta} \times \boldsymbol{\beta} \\ &= -(\boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} + \boldsymbol{\nu} \operatorname{div} \boldsymbol{\nu} + \boldsymbol{\beta} \operatorname{div} \boldsymbol{\beta}) = [\boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{S}(\boldsymbol{\nu}) + \boldsymbol{S}(\boldsymbol{\beta})]/2.\end{aligned}$$

Here $\boldsymbol{K}_\tau = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} = k\boldsymbol{\nu}$, $\boldsymbol{K}_\nu = (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\nu} = \operatorname{rot} \boldsymbol{\nu} \times \boldsymbol{\nu}$, and $\boldsymbol{K}_\beta = (\boldsymbol{\beta} \cdot \nabla)\boldsymbol{\beta} = \operatorname{rot} \boldsymbol{\beta} \times \boldsymbol{\beta}$ are the curvature vectors of the vector lines L_τ, L_ν, and L_β of the fields $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$, respectively. Then, in D, $\boldsymbol{S}^* = \boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau} \times \boldsymbol{R}^*$, where the vector field $\boldsymbol{R}^*$ is expressed by any of formulas (4)–(6)*.*

Proof. The expression $\boldsymbol{S}^* = -(\boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} + \boldsymbol{\nu} \operatorname{div} \boldsymbol{\nu} + \boldsymbol{\beta} \operatorname{div} \boldsymbol{\beta})$ follows from the well-known formulas $\operatorname{div}(\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b} \cdot \operatorname{rot} \boldsymbol{a}) - (\boldsymbol{a} \cdot \operatorname{rot} \boldsymbol{b})$, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$, $\boldsymbol{\tau} = \boldsymbol{\nu} \times \boldsymbol{\beta}$ [7], $\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}$, and $\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$. Combining this expression for $\boldsymbol{S}^*$ with the original one (in terms of rotors), we obtain $\boldsymbol{S}^* = [\boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{S}(\boldsymbol{\nu}) + \boldsymbol{S}(\boldsymbol{\beta})]/2$. Substituting the formulas for $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ from Lemma 1, the formula for k^2 from Lemma 2, the relations between $\boldsymbol{\tau}$, $\boldsymbol{\nu}_1$, and $\boldsymbol{\nu}_2$ from Lemma 1 into the expression for $\boldsymbol{S}^*$, after lengthy but simple calculations, we obtain $-\boldsymbol{S}^* = \operatorname{grad} \alpha \times \boldsymbol{k} + \operatorname{grad} \theta \times \boldsymbol{\nu}_2 + \operatorname{grad} \varphi \times \boldsymbol{\tau}$, where the function φ is defined in Lemma 2. Combining the latter equality with the first formula for $\boldsymbol{S}(\boldsymbol{\tau})$ from Lemma 1 and with allowance for $\sin\theta\, \boldsymbol{\nu}_1 = \cos\theta\, \boldsymbol{\tau} - \boldsymbol{k}$, we obtain $\boldsymbol{S}^* = \boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau} \times \boldsymbol{R}^*$, where $\boldsymbol{R}^* = \operatorname{grad} \varphi + \cos\theta \operatorname{grad} \alpha$.

We will now show that the latter vector $\boldsymbol{R}^*$ satisfies the invariant expression (4). Indeed, $\boldsymbol{R}^* \cdot \boldsymbol{\tau} = \varphi_s + \alpha_s \cos\theta = \varkappa$ by virtue of Lemma 2. Multiplying the identity $\boldsymbol{S}^* = \boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau} \times \boldsymbol{R}^*$ vectorially by $\boldsymbol{\nu}$ and by $\boldsymbol{\beta}$ and using the well-known formulas $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$, $\boldsymbol{a} \times \boldsymbol{a} = 0$, and $\boldsymbol{\tau} \cdot \boldsymbol{\nu} = \boldsymbol{\tau} \cdot \boldsymbol{\beta} = 0$ [7], we obtain $\boldsymbol{\nu} \cdot \boldsymbol{R}^* = -\operatorname{div} \boldsymbol{\beta}$ and $\boldsymbol{\beta} \cdot \boldsymbol{R}^* = k + \operatorname{div} \boldsymbol{\nu}$, respectively. This brings about the desired formula for $\boldsymbol{R}^*$. Using the formulas $k\boldsymbol{\beta} = \boldsymbol{\tau} \times (\operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau}) = \operatorname{rot} \boldsymbol{\tau} - \boldsymbol{\tau}(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau}) \Rightarrow k = \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\tau}$, $\operatorname{rot} \boldsymbol{\tau} = \operatorname{rot}(\boldsymbol{\nu} \times \boldsymbol{\beta}) = \boldsymbol{\nu} \operatorname{div} \boldsymbol{\beta} - \boldsymbol{\beta} \operatorname{div} \boldsymbol{\nu} + \nabla(\boldsymbol{\nu}, \boldsymbol{\beta})$, $\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$, $\operatorname{div} \boldsymbol{\beta} = \operatorname{div}(\boldsymbol{\tau} \times \boldsymbol{\nu}) = -\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\nu}$, and $\operatorname{div} \boldsymbol{\nu} = \operatorname{div}(\boldsymbol{\beta} \times \boldsymbol{\tau}) = \boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\beta} - k$, we obtain the remaining representations for $\boldsymbol{R}^*$ in the lemma. □

Remark 1. The formula $\operatorname{div} \boldsymbol{S}^* = (\varkappa - \boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau})^2 - [\boldsymbol{\tau} \cdot (\operatorname{rot} \boldsymbol{\nu} \times \operatorname{rot} \boldsymbol{\beta}) + \boldsymbol{\nu} \cdot (\operatorname{rot} \boldsymbol{\beta} \times \operatorname{rot} \boldsymbol{\tau}) + \boldsymbol{\beta} \cdot (\operatorname{rot} \boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\nu})]$ is proved in a similar way.

From formulas (2), $\boldsymbol{R}^* = \operatorname{grad} \varphi + \cos\theta \operatorname{grad} \alpha$, $\boldsymbol{S}^* = \boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau} \times \boldsymbol{R}^*$, and Lemma 1, we obtain

Corollary 1. *Under conditions* (A)–(C)*, in D, we have* $\operatorname{rot} \boldsymbol{R}^* = \sin\theta \boldsymbol{A} = \sin\theta\, (\operatorname{grad} \alpha \times \operatorname{grad} \theta)$*,* $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 2(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{R}^*)$*, and* $\operatorname{div} \boldsymbol{S}^* = \frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) + \varkappa(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau}) + k(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\beta})$*.*

The latter is derived using the equalities $\operatorname{div} \boldsymbol{S}^* = \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{R}^* \cdot \operatorname{rot} \boldsymbol{\tau} - \operatorname{rot} \boldsymbol{R}^* \cdot \boldsymbol{\tau}$ and $\boldsymbol{R}^* \cdot \operatorname{rot} \boldsymbol{\tau} = \varkappa(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau}) + k(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\beta})$ by virtue of $\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$, $\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\tau} = k$, and $\operatorname{rot} \boldsymbol{R}^* \cdot \boldsymbol{\tau} = \frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$.

2.2. Invariant forms of the vector rot $\boldsymbol{R}^*$

Theorem 1. *Let conditions* (A)–(C) *be satisfied. Then, the quantity* $\operatorname{rot} \boldsymbol{R}^*$ *with the vector field* $\boldsymbol{R}^*$ *defined by any one of formulas* (4)–(6) *has any of the representations*

$$\begin{aligned} \operatorname{rot} \boldsymbol{R}^* &= \frac{1}{2} \boldsymbol{\tau} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) - k\boldsymbol{\nu}(\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta}) - k\boldsymbol{\beta}(\varkappa + \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta}) && (9) \\ &= \boldsymbol{\tau} \Big[\frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) + k(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\beta})\Big] - k \operatorname{rot} \boldsymbol{\beta} - \varkappa k \boldsymbol{\beta} \\ &= \boldsymbol{\tau} \operatorname{div} \boldsymbol{S}^* - \varkappa \operatorname{rot} \boldsymbol{\tau} - k \operatorname{rot} \boldsymbol{\beta}, && (10) \end{aligned}$$

where the vector fields $\boldsymbol{S}(\boldsymbol{\tau})$ *and* $\boldsymbol{S}^*$ *are defined in* (7) *and in Lemma 3.*

Proof. We calculate the quantity $\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta}$ using the formulas $\boldsymbol{\beta} = (-\alpha_s \sin\theta\, \boldsymbol{\nu}_1 + \theta_s \boldsymbol{\nu}_2)/k$ (from Lemma 1), $\operatorname{rot} \boldsymbol{\nu}_1 = \cos\theta\, (\operatorname{grad} \alpha \times \boldsymbol{\nu}_2) - \operatorname{grad} \theta \times \boldsymbol{\tau}$, and $\operatorname{rot} \boldsymbol{\nu}_2 = -\operatorname{grad} \alpha \times (\cos\alpha\, \boldsymbol{i} + \sin\alpha\, \boldsymbol{j})$, the relations $\boldsymbol{\tau} = \boldsymbol{\nu}_1 \times \boldsymbol{\nu}_2$, $\boldsymbol{\nu}_1 = \boldsymbol{\nu}_2 \times \boldsymbol{\tau}$, and $\boldsymbol{\nu}_2 = \boldsymbol{\tau} \times \boldsymbol{\nu}_1$, and the formulas of Lemma 2 for k^2 and $\varkappa$. Then we obtain $\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta} = -\varkappa + \sin\theta\, [\theta_s(\operatorname{grad} \alpha \cdot \boldsymbol{\nu}) - \alpha_s(\operatorname{grad} \theta \cdot \boldsymbol{\nu})]/k$. Here, substituting $\alpha_s = \operatorname{grad} \alpha \cdot \boldsymbol{\tau}$, $\theta_s = \operatorname{grad} \theta \cdot \boldsymbol{\tau}$ and using the well-known formula $(\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \cdot \boldsymbol{d}) = [\boldsymbol{a} \times \boldsymbol{b}] \cdot [\boldsymbol{c} \times \boldsymbol{d}]$ [7, § 7] for $\boldsymbol{a} = \operatorname{grad} \theta$, $\boldsymbol{b} = \operatorname{grad} \alpha$, $\boldsymbol{c} = \boldsymbol{\tau}$, and $\boldsymbol{d} = \boldsymbol{\nu}$, we obtain $\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta} = -\varkappa - \sin\theta\, [(\boldsymbol{\tau} \times \boldsymbol{\nu}) \cdot (\operatorname{grad} \alpha \times \operatorname{grad} \theta)] = -\varkappa - (\boldsymbol{\beta} \cdot \sin\theta \boldsymbol{A})/k = -\varkappa - (\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{R}^*)/k$. This results in $\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{R}^* = -k[\varkappa + \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta}]$. Similarly we obtain $\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta} = -\sin\theta\, [\theta_s(\operatorname{grad} \alpha \cdot \boldsymbol{\beta}) - \alpha_s(\operatorname{grad} \theta \cdot \boldsymbol{\beta})] = -(\boldsymbol{\nu} \cdot \sin\theta \boldsymbol{A})/k = -(\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{R}^*)/k$, whence $\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{R}^* = -k\boldsymbol{\nu}(\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta})$. From Corollary 1 we have $\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{R}^* = \frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$, which leads to formula (9). From this formula, by virtue of Corollary 1, we obtain identity (10). □

In the similar way we prove the following

Lemma 4. *Let conditions* (A)–(C) *be satisfied. Then the vector fields* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *defined in Lemma 1 are expressed in the domain* D *in terms of the characteristics* $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, k, *and* $\varkappa$ *of the curves* $L_\tau \in \{L_\tau\}$ *by the formulas* $\boldsymbol{A}_1 = k\boldsymbol{\nu}(\varkappa + \boldsymbol{\beta}\cdot\operatorname{rot}\boldsymbol{\beta}) - k\boldsymbol{\beta}(\boldsymbol{\nu}\cdot\operatorname{rot}\boldsymbol{\beta})$ *and* $\boldsymbol{A}_2 = k[k\boldsymbol{\tau} + \boldsymbol{\nu}(\boldsymbol{\beta}\cdot\operatorname{rot}\boldsymbol{\nu}) - \boldsymbol{\beta}(\varkappa + \boldsymbol{\nu}\cdot\operatorname{rot}\boldsymbol{\nu})]$.

2.3. The relationship between the quantities $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = -2\boldsymbol{K}$, $\varkappa$, $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$. On the conservation laws for the family of curves $\boldsymbol{L_\tau}$

Theorem 2. *Let conditions* (A)–(C) *be satisfied. Then, in the domain* D, *we have* $\frac{1}{2}\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = \varkappa(\varkappa - \boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau}) - \boldsymbol{\tau}\cdot(\operatorname{rot}\boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\beta}) \Leftrightarrow \varkappa^2 = \frac{1}{2}\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) + \varkappa(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau}) + \boldsymbol{\tau}\cdot(\operatorname{rot}\boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\beta}) = \boldsymbol{\tau}\cdot(\operatorname{rot}\boldsymbol{R}^* + \varkappa\operatorname{rot}\boldsymbol{\tau} + \operatorname{rot}\boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\beta})$.

Proof. From the definition of the quantities $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ in Lemma 1 we obtain $\sin\theta\operatorname{grad}\alpha = (\theta_s\boldsymbol{A}_1 + \alpha_s\sin\theta\boldsymbol{A}_2)/k^2$ and $\operatorname{grad}\theta = (-\alpha_s\sin\theta\boldsymbol{A}_1 + \theta_s\boldsymbol{A}_2)/k^2 \Rightarrow \sin\theta\boldsymbol{A} = \sin\theta\operatorname{grad}\alpha\times\operatorname{grad}\theta = \operatorname{rot}\boldsymbol{R}^* = (\boldsymbol{A}_1\times\boldsymbol{A}_2)/k^2$, whence, using the formulas from Lemma 4, we have

$$\begin{aligned}\operatorname{rot}\boldsymbol{R}^* = \boldsymbol{\tau}[\varkappa^2 - \varkappa(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau}) - \boldsymbol{\tau}\cdot(\operatorname{rot}\boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\beta})] -\\ k\boldsymbol{\nu}(\boldsymbol{\nu}\cdot\operatorname{rot}\boldsymbol{\beta}) - k\boldsymbol{\beta}(\varkappa + \boldsymbol{\beta}\cdot\operatorname{rot}\boldsymbol{\beta}).\end{aligned} \tag{11}$$

The theorem is proved by multiplying the latter equation by $\boldsymbol{\tau}$ and using formula (9). □

Remark 2. The formulas in Corollary 1 and Theorem 2 containing the expressions $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$ and the formulas in Theorem 1 are respectively the scalar and vector analogs to the conservation law $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) = 0 \Leftrightarrow \operatorname{div}\boldsymbol{S}_p^* = 0$ of the plane case for the family of plane curves $\{L_\tau\}$. In the plane case, we have $\boldsymbol{\tau} = \boldsymbol{\tau}(x,y)$, $\boldsymbol{\beta} \equiv \boldsymbol{k}$, $\varkappa = 0 \Rightarrow \boldsymbol{R}^* = 0$, $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{S}_p^*$, and $\operatorname{rot}\boldsymbol{R}^* = 0$, and these formulas imply this conservation law. In the three-dimensional case, Theorem 1 leads to a higher-order conservation law $\operatorname{div}\boldsymbol{F} = 0$ for the family $\{L_\tau\}$ of curves L_τ. Here the vector solenoidal field $\boldsymbol{F}$ is expressed in terms of the characteristics $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, k, and $\varkappa$ of the curves L_τ and is the right-hand side of any of formulas (9)–(11). For example,

$$\operatorname{div}\left[\frac{1}{2}\boldsymbol{\tau}\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau}) - k\boldsymbol{\nu}(\boldsymbol{\nu}\cdot\operatorname{rot}\boldsymbol{\beta}) - k\boldsymbol{\beta}(\varkappa + \boldsymbol{\beta}\cdot\operatorname{rot}\boldsymbol{\beta})\right] = 0, \tag{12}$$

$$\operatorname{div}[\boldsymbol{\tau}\operatorname{div}\boldsymbol{S}^* - \varkappa\operatorname{rot}\boldsymbol{\tau} - k\operatorname{rot}\boldsymbol{\beta}] = 0, \tag{13}$$

where the fields $\boldsymbol{S}(\boldsymbol{\tau})$, $\boldsymbol{S}^*$ are expressed using formulas (7) and Lemma 3.

2.4. Solenoidal representation of the vector $\boldsymbol{P}$ in terms of the field $\boldsymbol{R}^*$

Theorem 3. *Assume that for the family $\{L_\tau\}$ of streamlines L_τ of the unit vector field $\boldsymbol{\tau}$ in the domain D conditions* (A)–(C) *are satisfied and* $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$*, k, and $\varkappa$ are the Frenet basis, the first curvature, and the second curvature of the curves L_τ. Then the field $\boldsymbol{P}$ in formula* (1) *can be represented as* (3)*, where the field $\boldsymbol{R}^*$ is expressed by any of the invariant forms* (4)–(6)*. Furthermore, in addition to formula* (2)*, anyone of the expressions in terms of the quantities $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, k, and $\varkappa$ contained in the right-hand sides of formulas* (9)–(11) *is valid for the field* $(-\boldsymbol{P})$.

Proof. We show that the right-hand sides of (2) and (9) differ only in their signs. From the second Frenet equation $d\boldsymbol{\nu}/ds = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\nu} = -k\boldsymbol{\tau} + \varkappa\boldsymbol{\beta}$ and the formulas $\operatorname{rot}\boldsymbol{\beta} = \operatorname{rot}(\boldsymbol{\tau} \times \boldsymbol{\nu}) = (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\tau} - (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\nu} + \boldsymbol{\tau}\operatorname{div}\boldsymbol{\nu} - \boldsymbol{\nu}\operatorname{div}\boldsymbol{\tau}$, $\operatorname{div}\boldsymbol{\nu} = \boldsymbol{\tau} \cdot \operatorname{rot}\boldsymbol{\beta} - \boldsymbol{\beta} \cdot \operatorname{rot}\boldsymbol{\tau}$, and $k = \boldsymbol{\beta} \cdot \operatorname{rot}\boldsymbol{\tau}$, we obtain $(\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\tau} = \operatorname{rot}\boldsymbol{\beta} + \varkappa\boldsymbol{\beta} - \boldsymbol{\tau}(\boldsymbol{\tau} \cdot \operatorname{rot}\boldsymbol{\beta}) + \boldsymbol{\nu}\operatorname{div}\boldsymbol{\tau}$. Next we use $K = -\frac{1}{2}\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$ [2, Ch. 1, § 8], $(\boldsymbol{K}_\tau \cdot \nabla)\boldsymbol{\tau} = k(\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\tau}$, and the theorem is proved. □

Corollary 2. *Representation* (1) *is equivalent to the formula*

$$K = -\operatorname{grad}(\boldsymbol{r} \cdot \boldsymbol{\tau}) \cdot \operatorname{rot}\boldsymbol{R}^* \quad \Leftrightarrow \quad K = \operatorname{div}[\operatorname{grad}(\boldsymbol{r} \cdot \boldsymbol{\tau}) \times \boldsymbol{R}^*].$$

Remark 3. The Frenet unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ and the first curvature k of the curves L_τ can be expressed in terms of $\boldsymbol{\tau}$:

$$\boldsymbol{\nu} = (\operatorname{rot}\boldsymbol{\tau} \times \boldsymbol{\tau})/k, \quad \boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}, \quad k = |\operatorname{rot}\boldsymbol{\tau} \times \boldsymbol{\tau}|. \tag{14}$$

Substituting the latter for $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ into the formula [2, Ch. 1, § 15]:

$$\varkappa = \frac{1}{2}(\boldsymbol{\tau} \cdot \operatorname{rot}\boldsymbol{\tau} - \boldsymbol{\nu} \cdot \operatorname{rot}\boldsymbol{\nu} - \boldsymbol{\beta} \cdot \operatorname{rot}\boldsymbol{\beta}) \tag{15}$$

we also express the second curvature $\varkappa$ in terms of the single quantity $\boldsymbol{\tau}$.

As, by virtue of formulas (4)–(7), Lemmas 3 and 4, and Theorems 1–3, the quantities $\boldsymbol{S}(\boldsymbol{\tau})$, $\boldsymbol{S}^*$, $\boldsymbol{R}^*$, $\operatorname{div}\boldsymbol{S}(\boldsymbol{\tau})$, $\operatorname{div}\boldsymbol{S}^*$, $\operatorname{rot}\boldsymbol{R}^*$, and K are expressed in terms of the unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$, the first curvature k and the second curvature $\varkappa$ of the curves L_τ, it follows that all these quantities can ultimately be expressed only in terms of the field $\boldsymbol{\tau}$. Therefore, all the formulas in Theorems 1–3 and Lemmas 3 and 4 can be expressed only in terms of the field $\boldsymbol{\tau}$, i.e. the unit tangent vectors of the curves L_τ.

3. Properties of the family $\{S_\tau\}$ of surfaces S_τ

3.1. Conditions on the family $\{S_\tau\}$. Let us assume that for the field $\boldsymbol{\tau}$ in D, there exists a family of surfaces S_τ orthogonal to the field $\boldsymbol{\tau}$. According to the Jacobi theorem [2, Ch. 1, § 1], this is equivalent to the identity $\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$ in D. Let $\{S_\tau\}$ be the family of surfaces S_τ with a unit normal $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ which continuously fill the domain D in the space x, y, z. The principal direction will be represented by the unit vector $\boldsymbol{l}_i$ $(i = 1, 2)$ with the corresponding direction; the vector $\boldsymbol{l}_i$ is the unit tangent vector of the curvature line L_i on S_τ, and at a point $(x, y, z) \in S_\tau$ it is equal to the derivative of the radius vector $\boldsymbol{r} = \boldsymbol{r}(x, y, z)$ of the point of the surface S_τ with respect to the principal direction at the point (x, y, z). Suppose that:

(D) one and only one surface $S_\tau \in \{S_\tau\}$ passes through each point $(x, y, z) \in D$;

(E) at each point $(x, y, z) \in D$, there exists a right-hand system of mutually orthogonal unit vectors $\boldsymbol{\tau}$, $\boldsymbol{l}_1$, and $\boldsymbol{l}_2$, where $\boldsymbol{\tau}$ is the unit normal and $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are the principal directions on the surface S_τ passing through this point. For this, it is sufficient that each surface $S_\tau \in \{S_\tau\}$ be C^2-regular [8]. Thus, in the domain D, we define three mutually orthogonal unit vector fields $\boldsymbol{\tau}(x, y, z)$, $\boldsymbol{l}_1(x, y, z)$, and $\boldsymbol{l}_2(x, y, z)$; $\boldsymbol{l}_1 = \boldsymbol{l}_2 \times \boldsymbol{\tau}$, $\boldsymbol{l}_2 = \boldsymbol{\tau} \times \boldsymbol{l}_1$, and $\boldsymbol{\tau} = \boldsymbol{l}_1 \times \boldsymbol{l}_2$;

(F) $\boldsymbol{\tau}, \boldsymbol{l}_1, \boldsymbol{l}_2 \in C^1(D)$.

3.2. The equality of non-holonomicity values of the fields $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$

Theorem 4. *Let a family* $\{S_\tau\}$ *of surfaces* S_τ *with the unit normal* $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ *satisfy conditions* (D)–(F) *in the domain* D. *Then the non-holonomicity values of the vector fields of the principal directions* $\boldsymbol{l}_1$ *and* $\boldsymbol{l}_2$ (*unit tangent vectors of the curvature lines* L_i *on* S_τ) *are equal in* D :

$$\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_1 = \boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_2. \tag{16}$$

Proof. Writing down the general formulas [7, § 17] $\operatorname{rot}[\boldsymbol{a} \times \boldsymbol{b}] = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{a} \operatorname{div} \boldsymbol{b} - \boldsymbol{b} \operatorname{div} \boldsymbol{a}$, and $\operatorname{grad}(\boldsymbol{a} \cdot \boldsymbol{b}) = (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} + (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + \boldsymbol{b} \times \operatorname{rot} \boldsymbol{a} + \boldsymbol{a} \times \operatorname{rot} \boldsymbol{b}$ for $\boldsymbol{a} = \boldsymbol{l}_2$ and $\boldsymbol{b} = \boldsymbol{\tau}$, subtracting and taking into account the Rodrigues formulas [8] written as $(\boldsymbol{l}_1 \cdot \nabla)\boldsymbol{\tau} = -k_1 \boldsymbol{l}_1$ and $(\boldsymbol{l}_2 \cdot \nabla)\boldsymbol{\tau} = -k_2 \boldsymbol{l}_2$, we obtain $\operatorname{rot} \boldsymbol{l}_1 = (2k_2 + \operatorname{div} \boldsymbol{\tau})\boldsymbol{l}_2 - \boldsymbol{\tau} \operatorname{div} \boldsymbol{l}_2 + \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{l}_2 + \operatorname{rot} \boldsymbol{l}_2 \times \boldsymbol{\tau}$. Multiplying the latter scalarly by $\boldsymbol{l}_1$, we prove the theorem. Another proof follows from the fact that the principal curvatures are stationary values of the normal curvatures [9]. □

Remark 4. Theorem 2 for $\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$, $\boldsymbol{\tau} \in C^2(D)$ implies the following formula relating to the Gaussian curvature K of the surface $S_\tau \in \{S_\tau\}$, the Frenet basis $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$, and the second curvature $\varkappa$ of the vector lines L_τ of the field of normals $\boldsymbol{\tau}$ of the surfaces S_τ:

$$K = \boldsymbol{\tau} \cdot (\operatorname{rot} \boldsymbol{\nu} \times \operatorname{rot} \boldsymbol{\beta}) - \varkappa^2. \tag{17}$$

Here the second curvature $\varkappa$ of the curves L_τ can be expressed in terms of the Frenet unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$ by formula (15), assuming $\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$.

3.3. Geometric meaning of the field $\boldsymbol{S}(\boldsymbol{\tau})$

Theorem 5. *Let $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ be a unit vector field in the domain D; the family $\{L_\tau\}$ of vector lines L_τ of the field $\boldsymbol{\tau}$ and the family $\{S_\tau\}$ of surfaces S_τ with normal $\boldsymbol{\tau}$ are mutually orthogonal in D. Let conditions* (D)–(F) *be satisfied in D. Then the field $\boldsymbol{S}(\boldsymbol{\tau})$ of the form* (7) *at any point $(x, y, z) \in D$ is the sum of the three curvature vectors*: $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{K}_\tau + \boldsymbol{K}_{g1} + \boldsymbol{K}_{g2} = \boldsymbol{K}_\tau + 2H\boldsymbol{\tau}$ (*H is the mean curvature of the surface*). *Here $\boldsymbol{K}_\tau = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau}$ is the curvature vector of the vector line L_τ of the field $\boldsymbol{\tau}$ at the point (x, y, z); $\boldsymbol{K}_{g1} = k_{g1}\boldsymbol{\tau}$ and $\boldsymbol{K}_{g2} = k_{g2}\boldsymbol{\tau}$ are the curvature vectors* (*at the same point*) *of two geodesic lines with the curvatures k_{g1} and k_{g2} at the surface S_τ which pass through the point $(x, y, z) \in S_\tau$ in any two mutually orthogonal directions.*

Proof. For a geodesic line γ_i $(i = 1, 2)$ on the surface S_τ, the principal normal coincides with the normal $\boldsymbol{\tau}$ to the surface, and the curvature k_{gi} everywhere is equal to the normal curvature k_{ni} [1, § 73]. Therefore, $\boldsymbol{K}_{gi} = k_{ni}\boldsymbol{\tau} \Rightarrow \boldsymbol{K}_{g1} + \boldsymbol{K}_{g2} = (k_{1n} + k_{2n})\boldsymbol{\tau} = (k_1 + k_2)\boldsymbol{\tau} = 2H\boldsymbol{\tau} = -\boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau}$. □

4. Application of the obtained geometric formulas to the mathematical physics equations

Suppose that $\boldsymbol{v} = \boldsymbol{v}(x, y, z) = |\boldsymbol{v}|\boldsymbol{\tau}$ is a vector field with direction $\boldsymbol{\tau}$ ($|\boldsymbol{\tau}| \equiv 1$) and modulus $|\boldsymbol{v}| \neq 0$, defined in a domain D of the three-dimensional Euclidean space E^3. The vector lines L_τ of the fields $\boldsymbol{v}$ and $\boldsymbol{\tau}$ coincide and $\boldsymbol{v} \cdot \operatorname{rot} \boldsymbol{v} = |\boldsymbol{v}|^2(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau})$. Therefore, a necessary and sufficient condition for the existence of a family $\{S_\tau\}$ of surfaces S_τ with the unit normal $\boldsymbol{\tau}$, which are orthogonal to the fields $\boldsymbol{v}$ and $\boldsymbol{\tau}$ (the condition on holonomicity of the field $\boldsymbol{\tau}$) is the identity $\boldsymbol{v} \cdot \operatorname{rot} \boldsymbol{v} = 0 \Leftrightarrow \boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$.

Remark 5. On the other hand, as stated by the theorem from Problem 136 in [7, § 17], in order that the variable vector field $\boldsymbol{v}$ can be represented as $\boldsymbol{v} = \psi \operatorname{grad} \varphi$, where φ and ψ are scalar functions, it is also necessary and sufficient that the identity $\boldsymbol{v} \cdot \operatorname{rot} \boldsymbol{v} = 0$ be satisfied. Therefore, for vector fields of the form $\boldsymbol{v} = \psi \operatorname{grad} \varphi$, and only for them, there exists a family $\{S_\tau\}$ of surfaces S_τ orthogonal to the fields $\boldsymbol{v}$ and $\boldsymbol{\tau}$, i.e., the case of the holonomic field of directions $\boldsymbol{\tau}$ reduces to fields of the form $\boldsymbol{v} = \psi \operatorname{grad} \varphi$. Below we will take into account the fact that $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{T}(\boldsymbol{v})$, where $\boldsymbol{T}(\boldsymbol{v}) = \operatorname{grad} \ln \boldsymbol{v} + (\operatorname{rot} \boldsymbol{v} \times \boldsymbol{v} - \boldsymbol{v} \operatorname{div} \boldsymbol{v})/|\boldsymbol{v}|^2$ (see Section 1).

4.1. The eikonal equation. We consider the eikonal equation $|\operatorname{grad} \tau|^2 \stackrel{\text{def}}{=} \tau_x^2 + \tau_y^2 + \tau_z^2 = n^2(x,y,z)$ for a scalar time field $\tau = \tau(x,y,z)$, which is the basic mathematical model of kinematic seismics (geometric optic) in an inhomogeneous isotropic medium with the refractive index $n(x,y,z)$. The function $\tau(x,y,z)$ is the travel time of a signal (wave) of any nature, whose kinematics satisfies the Fermat principle, along the ray (the geodesic line of the metric $ds^2 = n^2(x,y,z)(dx^2+dy^2+dz^2)$) which connects the point source and the point (x,y,z). In this case, the ray plays the role of a curve L_τ and is the vector line of the vector potential (non-force) field $\boldsymbol{v} = \operatorname{grad} \tau$ with tangent unit vector $\boldsymbol{\tau} = \operatorname{grad} \tau / n$ and modulus $|\operatorname{grad} \tau| = n$. Obviously, in this case, the field $\boldsymbol{\tau}$ is holonomic ($\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau} = \boldsymbol{v} \cdot \operatorname{rot} \boldsymbol{v} = 0$); the role of the surfaces S_τ orthogonal to the rays L_τ is played by the wavefronts $\tau(x,y,z) = \text{const}$ (level surfaces of the scalar field τ).

From the general geometric formula (17) for $\tau \in C^3(D)$ and $n \in C^2(D)$, we obtain the following expression for the Gaussian curvature K of the front S_τ: $K = \boldsymbol{\tau} \cdot (\operatorname{rot} \boldsymbol{\nu} \times \operatorname{rot} \boldsymbol{\beta}) - \varkappa^2$. Here $\boldsymbol{\tau} = \operatorname{grad} \tau / n$; the principal normal $\boldsymbol{\nu}$ and the binormal $\boldsymbol{\beta}$ of the ray are calculated by formulas (14), and the second curvature $\varkappa$ of the ray is calculated by (15). Another formula for K of the form $K = -\frac{1}{2} \operatorname{div} \boldsymbol{T}$, where $\boldsymbol{T} = \boldsymbol{T}(\operatorname{grad} \tau) = \operatorname{grad} \ln n - \frac{\Delta \tau}{n^2} \operatorname{grad} \tau$, or $K = -\frac{1}{2}\Big[\Delta \ln n - \operatorname{div}\Big(\frac{\Delta \tau}{n^2} \operatorname{grad} \tau\Big)\Big]$, follows from the equality $K = -\frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$ [2, Ch. 1, § 8] and the identity $\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{T}(\boldsymbol{v})$. Generally, for the solutions τ of the eikonal equation, all the formulas in Sections 2 and 3 hold, including the formulas for $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$ and $\operatorname{div} \boldsymbol{S}^*$ from Corollary 1 and Theorem 2, which are analogs to the conservation law $\operatorname{div} \boldsymbol{T} = 0$ for the plane case (see [6] and Remark 2) and the conservation law (12), (13), with consideration of the equalities $\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$, $\boldsymbol{\tau} = \operatorname{grad} \tau / n$ and Remark 3.

4.2. The Euler hydrodynamic equations. Let $\boldsymbol{v} = \boldsymbol{v}(x,y,z,t) = v\boldsymbol{\tau}$ be the velocity in the Euler hydrodynamic equations $\boldsymbol{v}_t + \operatorname{grad} v^2/2 - \boldsymbol{v} \times \operatorname{rot} \boldsymbol{v} = \boldsymbol{F} - \operatorname{grad} p/\rho$, which can be rewritten as $\boldsymbol{G} = -\boldsymbol{T}(\boldsymbol{v})$ $(= -\boldsymbol{S}(\boldsymbol{\tau}))$, where $\boldsymbol{G} \stackrel{\text{def}}{=} (\boldsymbol{v}_t + \boldsymbol{v} \operatorname{div} \boldsymbol{v} + \operatorname{grad} p/\rho - \boldsymbol{F})/v^2$, in domain D; $v \stackrel{\text{def}}{=} |\boldsymbol{v}|$, $\boldsymbol{v} \in C^2(D)$, the pressure $p \in C^2(D)$, the density $\rho \in C^1(D)$, and the body force $\boldsymbol{F} \in C^1(D)$.

Here the role of the curves L_τ is played by the streamlines (the vector lines of the field $\boldsymbol{v}$ or $\boldsymbol{\tau}$ at fixed t). The class of fields $\boldsymbol{v}$ for which there exists a family $\{S_\tau\}$ of the surfaces S_τ orthogonal to the field $\boldsymbol{v}$ (curves L_τ)

is described by the formula $\boldsymbol{v} = \psi \operatorname{grad} \varphi$, where ψ and φ are scalar functions (Remark 5). This class, in particular, includes the potential field $\boldsymbol{v} = \operatorname{grad} \varphi$. To calculate the Gaussian curvature K of the surface S_τ orthogonal to the streamlines L_τ, we use formula (17) taking into account the equality $\boldsymbol{\tau} = \boldsymbol{v}/v$ and Remark 3. The second formula for K of the form $K = \frac{1}{2} \operatorname{div} \boldsymbol{G}$ follows from the identities $K = -\frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$ and $\boldsymbol{G} = -\boldsymbol{T}(\boldsymbol{v}) = -\boldsymbol{S}(\boldsymbol{\tau})$. The formulas for $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$ and $\operatorname{div} \boldsymbol{S}^*$ from Corollary 1 and Theorem 2, which are analogs to the conservation law $\operatorname{div} \boldsymbol{G} = 0$ for the plane case ([5] and Remark 2) and conservation laws (12), (13), also hold for the velocity field $\boldsymbol{v}$ with allowance for the equality $\boldsymbol{\tau} = \boldsymbol{v}/v$ and Remark 3.

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