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On the conservation laws for a family of surfaces

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Abstract. A family $\{S_\tau\}$ of surfaces S_τ with the unit normal $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ in the Euclidean space E^3 is considered. The surfaces S_τ continuously fill a domain D in E^3. For the family $\{S_\tau\}$ of surfaces S_τ, the law of conservation $\operatorname{div} \boldsymbol{F} = 0$ is proved, where the solenoidal vector field $\boldsymbol{F}$ is expressed in terms of the main classical characteristics of the surfaces S_τ: the unit normal, the principal directions, the principal curvatures, the mean curvature, and the Gaussian curvature.

Keywords: vector field, family of surfaces, conservation law.

1. Introduction

This paper is a sequel to the previous publications [1, 2].

In mathematical physics, one sometimes has to deal with a family $\{S_\tau\}$ of surfaces S_τ with the unit normal $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ which are related to solutions of differential equations and continuously fill the domain in question. For example, for solutions τ of the eikonal equation $\tau_x^2 + \tau_y^2 + \tau_z^2 = n^2(x, y, z)$ (where $\tau = \tau(x, y, z)$ is the scalar time field and n is the refractive index), which is the basic mathematical model in kinematic seismics (geometric optics), the role of the surfaces S_τ is played by the wavefronts $\tau(x, y, z) =$ const. The curves L_τ orthogonal to the surfaces S_τ and having the unit tangent vector $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$) also form a family (the family of curves $\{L_\tau\}$) and continuously fill the domain under consideration. The curves L_τ are vector lines of the physical vector fields corresponding to the solutions of the equations of mathematical physics. For example, for the eikonal equation, the role of the curves L_τ is played by rays — the vector lines of the field $\boldsymbol{v} = \operatorname{grad} \tau = n\boldsymbol{\tau}$. Therefore, in this paper, we do not study the properties of individual curves and surfaces, but the properties of their families $\{L_\tau\}$ and $\{S_\tau\}$.

The basic characteristics of the curves L_τ of classical differential geometry [2–4] are the Frenet basis $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$, where $\boldsymbol{\tau}$ is the unit tangent vector, $\boldsymbol{\nu}$ is the principal normal, and $\boldsymbol{\beta}$ is the binormal, the first curvature k, and the second curvature $\varkappa$ being defined at each point of a given curve. The most important classical characteristics of the surface are its unit normal $\boldsymbol{\tau}$, the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, the principal curvatures k_1 and k_2, the mean curvature $H \stackrel{\text{def}}{=} (k_1 + k_2)/2$, and the Gaussian curvature $K \stackrel{\text{def}}{=} k_1 k_2$, which are defined at each point of a given surface. For the families $\{L_\tau\}$

and $\{S_\tau\}$, all the quantities $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, k, $\varkappa$ and $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, H, and K are the vector and the scalar fields in the domain D continuously filled with the curves L_τ and the surfaces S_τ. The symbols $\boldsymbol{a}\cdot\boldsymbol{b}$ and $\boldsymbol{a}\times\boldsymbol{b}$ denote the scalar and vector products of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, ∇ is the Hamiltonian operator, $(\boldsymbol{v}\cdot\nabla)\boldsymbol{a}$ is the derivative of the vector $\boldsymbol{a}$ in the direction of the vector $\boldsymbol{v}$.

In Section 2.3 of paper [1] and in [6], the conservation laws for a family of curves were obtained in the form of the identity $\operatorname{div}\boldsymbol{F} = 0$, where the vector field $\boldsymbol{F}$ is expressed in terms of the characteristics of the curves i.e. their Frenet basis vectors, first curvature, and second curvature.

In this paper, we prove the conservation law for a family $\{S_\tau\}$ of surfaces S_τ, i.e., a divergent identity of the form $\operatorname{div}\boldsymbol{F} = 0$, where the vector field $\boldsymbol{F}$ is expressed in terms of the basic characteristics of the surfaces S_τ: the quantities $\boldsymbol{\tau}$, $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, H, and K. (Generally, by the conservation law for a given mathematical object is meant a differential identity of the form $\operatorname{div}\boldsymbol{F} = 0$, where the vector field $\boldsymbol{F}$ is expressed in terms of the characteristics of this object. This definition agrees with the well-known concept of conservation law for a differential equation E [7], where the field $\boldsymbol{F}$ is expressed in terms of the solution to the equation E, the derivatives of this solution, and the parameters of the equation. An example is the conservation law $\operatorname{div}\boldsymbol{v} = 0$ for an ideal incompressible fluid, where $\boldsymbol{v}$ is the velocity [8].)

2. Conditions on the family of surfaces $\{S_\tau\}$ and on the family of curves $\{L_\tau\}$ orthogonal to $\{S_\tau\}$

Consider a domain D in the Euclidean space E^3 with the Cartesian coordinates x, y, z; $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors along the axes x, y, z; $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z) = \tau_1\boldsymbol{i} + \tau_2\boldsymbol{j} + \tau_3\boldsymbol{k}$ is the unit vector field defined in D, $\tau_k = \tau_k(x, y, z)$ are the scalar functions ($k = 1, 2, 3$), $|\boldsymbol{\tau}|^2 = 1$; L_τ is a vector line of the field $\boldsymbol{\tau}$ (with the unit tangent vector $\boldsymbol{\tau}$).

Let $\{L_\tau\}$ be a family of curves L_τ which continuously fill the domain D, and

(A) one and only one curve $L_\tau \in \{L_\tau\}$ passes through each point $(x, y, z) \in D$;

(B) at each point (x, y, z) of any curve $L_\tau \in \{L_\tau\}$, there exists a right-hand Frenet basis $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$, so that three mutually orthogonal vector fields $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ are defined in D, and $\boldsymbol{\tau} = \boldsymbol{\nu}\times\boldsymbol{\beta}$, $\boldsymbol{\nu} = \boldsymbol{\beta}\times\boldsymbol{\tau}$, $\boldsymbol{\beta} = \boldsymbol{\tau}\times\boldsymbol{\nu}$;

(C) $\boldsymbol{\tau} \in C^2(D)$.

In the domain D, let there exist a family of surfaces S_τ orthogonal to the family of curves $\{L_\tau\}$, i.e., to the field $\boldsymbol{\tau}$. According to the Jacobi theorem [9, Ch. 1, § 1], this is equivalent to the identity $\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau} = 0$ in D.

Therefore, $\{L_\tau\}$ is the family of vector lines of the field of normals $\boldsymbol{\tau}$ to the surfaces S_τ. Let $\{S_\tau\}$ be a family of surfaces S_τ with the unit normal $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ which continuously fill the domain D in the space of variables x, y, z. The principal direction will be represented by the unit vector $\boldsymbol{l}_i$ $(i = 1, 2)$ with the corresponding direction; the vector $\boldsymbol{l}_i$ is the unit tangent vector of the curvature line L_i on S_τ, and at a point $(x, y, z) \in S_\tau$ it is equal to the derivative of the radius vector $\boldsymbol{r} = \boldsymbol{r}(x, y, z)$ of the point of the surface S_τ in the principal direction at the point (x, y, z).

(D) Let one and only one surface $S_\tau \in \{S_\tau\}$ pass through each point $(x, y, z) \in D$.

(E) At each point $(x, y, z) \in D$, let there exist a right-hand system of mutually orthogonal unit vectors $\boldsymbol{\tau}$, $\boldsymbol{l}_1$, and $\boldsymbol{l}_2$, where $\boldsymbol{\tau}$ is the unit normal and $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are the principal directions at the surface S_τ passing through this point. For this, it is sufficient that each surface $S_\tau \in \{S_\tau\}$ be C^2-regular [4]. Thus, in the domain D, we have defined three mutually orthogonal unit vector fields $\boldsymbol{\tau}(x, y, z)$, $\boldsymbol{l}_1(x, y, z)$, and $\boldsymbol{l}_2(x, y, z)$; $\boldsymbol{l}_1 = \boldsymbol{l}_2 \times \boldsymbol{\tau}$, $\boldsymbol{l}_2 = \boldsymbol{\tau} \times \boldsymbol{l}_1$, $\boldsymbol{\tau} = \boldsymbol{l}_1 \times \boldsymbol{l}_2$;

(F) $\boldsymbol{\tau} \in C^2(D)$, $\boldsymbol{l}_1, \boldsymbol{l}_2 \in C^1(D)$.

Remark 1. As the initial object it is possible to take the family $\{S_\tau\}$ of surfaces S_τ with properties(D)–(F) which has the unit normal vector field $\boldsymbol{\tau}$ and to define the curves L_τ as the vector lines of this field $\boldsymbol{\tau}$. Obviously, the families $\{S_\tau\}$ and $\{L_\tau\}$ are mutually orthogonal.

3. Subsidiary propositions

We introduce the vector field

$$\boldsymbol{S}(\boldsymbol{\tau}) \stackrel{\text{def}}{=} \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} = \boldsymbol{K}_\tau - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau}, \tag{1}$$

where $\boldsymbol{K}_\tau = k\boldsymbol{\nu} = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = \dfrac{d\boldsymbol{\tau}}{ds} = \boldsymbol{\tau}_s$ is the curvature vector of the curve L_τ with the unit tangent vector $\boldsymbol{\tau}$ and the principal normal $\boldsymbol{\nu}$, L_τ is a streamline or a vector line of the field $\boldsymbol{\tau}$, k is its first curvature, d/ds is the differentiation operator with respect to the natural parameter s in the direction of $\boldsymbol{\tau}$ (along the curve L_τ).

Lemma 1 [1]. *Let a family $\{L_\tau\}$ of curves L_τ with the Frenet basis vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$, the first curvature k, and the second curvature $\varkappa$ in the domain D satisfy conditions* (A)–(C). *Let the field $\boldsymbol{S}^*$ be the sum of the three curvature vectors*:

$$\begin{aligned}\boldsymbol{S}^* &\stackrel{\text{def}}{=} \boldsymbol{K}_\tau + \boldsymbol{K}_\nu + \boldsymbol{K}_\beta = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} + (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{\nu} + (\boldsymbol{\beta}\cdot\nabla)\boldsymbol{\beta}\\ &= \operatorname{rot}\boldsymbol{\tau}\times\boldsymbol{\tau} + \operatorname{rot}\boldsymbol{\nu}\times\boldsymbol{\nu} + \operatorname{rot}\boldsymbol{\beta}\times\boldsymbol{\beta}\\ &= -(\boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\nu}\operatorname{div}\boldsymbol{\nu} + \boldsymbol{\beta}\operatorname{div}\boldsymbol{\beta}) = [\boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{S}(\boldsymbol{\nu}) + \boldsymbol{S}(\boldsymbol{\beta})]/2. \end{aligned} \tag{2}$$

Here $\boldsymbol{K}_\tau = (\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} = \operatorname{rot}\boldsymbol{\tau}\times\boldsymbol{\tau} = k\boldsymbol{\nu}$, $\boldsymbol{K}_\nu = (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{\nu} = \operatorname{rot}\boldsymbol{\nu}\times\boldsymbol{\nu}$, *and* $\boldsymbol{K}_\beta = (\boldsymbol{\beta}\cdot\nabla)\boldsymbol{\beta} = \operatorname{rot}\boldsymbol{\beta}\times\boldsymbol{\beta}$ *are the curvature vectors of the vector lines* L_τ, L_ν, *and* L_β *of the fields* $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, *and* $\boldsymbol{\beta}$, *respectively. Then, in* D,

$$\boldsymbol{S}^* = \boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau}\times\boldsymbol{R}^*, \tag{3}$$

where the vector field $\boldsymbol{R}^*$ *is represented by any of the formulas*

$$\boldsymbol{R}^* \stackrel{\text{def}}{=} \varkappa\boldsymbol{\tau} + k\boldsymbol{\beta} + \boldsymbol{\beta}\operatorname{div}\boldsymbol{\nu} - \boldsymbol{\nu}\operatorname{div}\boldsymbol{\beta}, \tag{4}$$

$$\boldsymbol{R}^* = \boldsymbol{\Phi} + \boldsymbol{S}^*\times\boldsymbol{\tau}, \tag{5}$$

$$\boldsymbol{R}^* = \varkappa\boldsymbol{\tau} + (\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\nu})\boldsymbol{\nu} + (\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\beta})\boldsymbol{\beta}, \tag{6}$$

$$\boldsymbol{R}^* = (\varkappa - \boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau})\boldsymbol{\tau} + \nabla(\boldsymbol{\nu},\boldsymbol{\beta}). \tag{7}$$

Here $\boldsymbol{\Phi} \stackrel{\text{def}}{=} \varkappa\boldsymbol{\tau} + k\boldsymbol{\beta}$ *is the Darboux vector* [10], $\nabla(\boldsymbol{\nu},\boldsymbol{\beta}) \stackrel{\text{def}}{=} (\boldsymbol{\beta}\cdot\nabla)\boldsymbol{\nu} - (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{\beta}$ *is the Poisson bracket* [9] *for* $\boldsymbol{\nu}$ *and* $\boldsymbol{\beta}$.

Thus, Lemma 1 determines the relation between the fields $\boldsymbol{S}^*$ and $\boldsymbol{S}(\boldsymbol{\tau})$; the vector field $\boldsymbol{R}^*$ is a measure of a difference between $\boldsymbol{S}^*$ and $\boldsymbol{S}(\boldsymbol{\tau})$. In [2], the following theorem on the relationship between the characteristics $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ of surfaces $S_\tau \in \{S_\tau\}$ and the characteristics $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ of the curves L_τ orthogonal to S_τ was obtained.

Theorem 1. *Let the family* $\{S_\tau\}$ *of surfaces* S_τ *with the unit normal* $\boldsymbol{\tau} = \boldsymbol{\tau}(x,y,z)$ *satisfy conditions* (D)–(F) *and let the family* $\{L_\tau\}$ *of curves* L_τ *orthogonal to* $\{S_\tau\}$ *satisfy conditions* (A)–(C). *Then at each point* $(x,y,z)\in D$, *the principal directions* $\boldsymbol{l}_1$ *and* $\boldsymbol{l}_2$ *of the surface* S_τ *passing through this point are expressed in terms of the Frenet basis vectors* $\boldsymbol{\nu}$ *and* $\boldsymbol{\beta}$ *of the curves* L_τ *according to the formulas*

$$\boldsymbol{l}_1 = \boldsymbol{\nu}\cos\omega + \boldsymbol{\beta}\sin\omega, \quad \boldsymbol{l}_2 = -\boldsymbol{\nu}\sin\omega + \boldsymbol{\beta}\cos\omega, \tag{8}$$

where $\omega = \omega(x,y,z)$ *is a scalar function* (ω *is the angle between the vectors* $\boldsymbol{l}_1$ *and* $\boldsymbol{\nu}$ *or between* $\boldsymbol{l}_2$ *and* $\boldsymbol{\beta}$). *In addition, the fields of the principal directions* $\boldsymbol{l}_1$ *and* $\boldsymbol{l}_2$ *in the domain* D *satisfy the identity*

$$\boldsymbol{l}_1\cdot\operatorname{rot}\boldsymbol{l}_1 = \boldsymbol{l}_2\cdot\operatorname{rot}\boldsymbol{l}_2. \tag{9}$$

In terms of the geometry of vector fields [9, Ch. 1, § 1], *identity* (9) *implies that the non-holonomicity values of the vector fields of the principal directions* $\boldsymbol{l}_1$ *and* $\boldsymbol{l}_2$ *are equal in* D. *Identity* (9) *is equivalent to the condition*

$$\operatorname{tg} 2\omega = -\frac{A}{B}, \tag{10}$$

in D, which defines the function ω in terms of $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$. Here $A \stackrel{\text{def}}{=} \boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\nu} - \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta}$ and $B \stackrel{\text{def}}{=} \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\nu} + \boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta}$. The principal curvatures k_1 and k_2 of the surfaces S_τ are given by the formulas

$$k_1 = -\operatorname{rot} \boldsymbol{l}_1 \cdot \boldsymbol{l}_2, \quad k_2 = \operatorname{rot} \boldsymbol{l}_2 \cdot \boldsymbol{l}_1. \tag{11}$$

The following statement is an analog to Lemma 1 for a family of surfaces $\{S_\tau\}$.

Lemma 2. *Let the conditions of Theorem 1 be satisfied and let the field $\boldsymbol{S}^*_l$ be the sum of the three curvature vectors*:

$$\begin{aligned}
\boldsymbol{S}^*_l &\stackrel{\text{def}}{=} \boldsymbol{K}_\tau + \boldsymbol{K}_1 + \boldsymbol{K}_2 = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} + (\boldsymbol{l}_1 \cdot \nabla)\boldsymbol{l}_1 + (\boldsymbol{l}_2 \cdot \nabla)\boldsymbol{l}_2 && (12)\\
&= \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} + \operatorname{rot} \boldsymbol{l}_1 \times \boldsymbol{l}_1 + \operatorname{rot} \boldsymbol{l}_2 \times \boldsymbol{l}_2 && (13)\\
&= -(\boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} + \boldsymbol{l}_1 \operatorname{div} \boldsymbol{l}_1 + \boldsymbol{l}_2 \operatorname{div} \boldsymbol{l}_2) && (14)\\
&= \{\boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{S}(\boldsymbol{l}_1) + \boldsymbol{S}(\boldsymbol{l}_2)\}/2.
\end{aligned}$$

Here $\boldsymbol{K}_\tau = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} = k\boldsymbol{\nu}$ is the curvature vector of the vector line L_τ of the normal field $\boldsymbol{\tau}$ of the surfaces S_τ and $\boldsymbol{K}_i = (\boldsymbol{l}_i \cdot \nabla)\boldsymbol{l}_i = \operatorname{rot} \boldsymbol{l}_i \times \boldsymbol{l}_i$ is the curvature vector of the curvature line L_i on S_τ $(i = 1, 2)$. Then, in the domain D,

$$\boldsymbol{S}^*_l = \boldsymbol{S}^* + \boldsymbol{\tau} \times \operatorname{grad} w, \quad \boldsymbol{S}^*_l = \boldsymbol{S}(\boldsymbol{\tau}) + \boldsymbol{\tau} \times \boldsymbol{R}^*_l, \tag{15}$$

where the vector field $\boldsymbol{R}^*_l$ can be represented by any of the formulas

$$\boldsymbol{R}^*_l \stackrel{\text{def}}{=} \operatorname{grad} w + \boldsymbol{R}^*, \tag{16}$$

$$\boldsymbol{R}^*_l = \varkappa_l \boldsymbol{\tau} + k\boldsymbol{\beta} + \boldsymbol{S}^*_l \times \boldsymbol{\tau}, \tag{17}$$

$$\boldsymbol{R}^*_l = \varkappa_l \boldsymbol{\tau} + \operatorname{rot} \boldsymbol{\tau} - (\boldsymbol{l}_1 \operatorname{div} \boldsymbol{l}_2 - \boldsymbol{l}_2 \operatorname{div} \boldsymbol{l}_1), \tag{18}$$

$$\boldsymbol{R}^*_l = \varkappa_l \boldsymbol{\tau} + \boldsymbol{l}_1(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{l}_1) + \boldsymbol{l}_2(\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{l}_2), \tag{19}$$

$$\varkappa_l \stackrel{\text{def}}{=} -(\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_1 + \boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_2)/2 = -\boldsymbol{l}_i \cdot \operatorname{rot} \boldsymbol{l}_i, \quad i = 1, 2, \tag{20}$$

and the quantities $\boldsymbol{S}(\boldsymbol{\tau})$, $\boldsymbol{S}^*$, $\boldsymbol{R}^*$, and w are given by formulas (1)–(10).

Proof. Using the well-known formula $\operatorname{rot}(\varphi \boldsymbol{a}) = \varphi \operatorname{rot} \boldsymbol{a} + \operatorname{grad} \varphi \times \boldsymbol{a}$ [3], from equalities (8), we obtain

$$\begin{aligned}\operatorname{rot}\boldsymbol{l}_1 &= \cos w \operatorname{rot}\boldsymbol{\nu} + \sin w \operatorname{rot}\boldsymbol{\beta} + \operatorname{grad} w \times \boldsymbol{l}_2,\\ \operatorname{rot}\boldsymbol{l}_2 &= -\sin w \operatorname{rot}\boldsymbol{\nu} + \cos w \operatorname{rot}\boldsymbol{\beta} - \operatorname{grad} w \times \boldsymbol{l}_1.\end{aligned} \tag{21}$$

From this, using the well-known formulas $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$, $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$ [3] and $\boldsymbol{\tau} = \boldsymbol{l}_1\times\boldsymbol{l}_2$, we obtain $\boldsymbol{K}_1 + \boldsymbol{K}_2 = \operatorname{rot}\boldsymbol{l}_1\times\boldsymbol{l}_2 + \operatorname{rot}\boldsymbol{l}_2\times\boldsymbol{l}_2 = \operatorname{rot}\boldsymbol{\nu}\times\boldsymbol{\nu} + \operatorname{rot}\boldsymbol{\beta}\times\boldsymbol{\beta} + \boldsymbol{\tau}\times\operatorname{grad} w = \boldsymbol{K}_\nu + \boldsymbol{K}_\beta + \boldsymbol{\tau}\times\operatorname{grad} w$. In view of definitions (2) and (12) of the vectors $\boldsymbol{S}^*$ and $\boldsymbol{S}_l^*$ and identity (3), the latter equality brings about identities (15), where the vector field $\boldsymbol{R}_l^*$ is given by formula (16).

Using equalities (5) and $\boldsymbol{S}^*\times\boldsymbol{\tau} = \boldsymbol{S}_l^*\times\boldsymbol{\tau} + \boldsymbol{\tau}\times(\boldsymbol{\tau}\times\operatorname{grad} w)$, from (16) we obtain $\boldsymbol{R}_l^* = \operatorname{grad} w + \boldsymbol{R}^* = \operatorname{grad} w + \varkappa\boldsymbol{\tau} + k\boldsymbol{\beta} + \boldsymbol{S}^*\times\boldsymbol{\tau} = \operatorname{grad} w + \varkappa\boldsymbol{\tau} + k\boldsymbol{\beta} + \boldsymbol{S}_l^*\times\boldsymbol{\tau} + \boldsymbol{\tau}(\operatorname{grad} w\cdot\boldsymbol{\tau}) - \operatorname{grad} w = \boldsymbol{\tau}(\varkappa + \operatorname{grad} w\cdot\boldsymbol{\tau}) + k\boldsymbol{\beta} + \boldsymbol{S}_l^*\times\boldsymbol{\tau}$. Then we have the equality $\varkappa + \operatorname{grad} w\cdot\boldsymbol{\tau} = \varkappa_l$, where the quantity $\varkappa_l$ is given by formula (20). This equality follows from the well-known formula $\varkappa = \frac{1}{2}(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau} - \boldsymbol{\nu}\cdot\operatorname{rot}\boldsymbol{\nu} - \boldsymbol{\beta}\cdot\operatorname{rot}\boldsymbol{\beta})$ [9, Ch. 1, § 15], given the identity $\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau} = 0$, equalities (8) and (9), and the formulas

$$\begin{aligned}\operatorname{rot}\boldsymbol{\nu} &= \cos w \operatorname{rot}\boldsymbol{l}_1 - \sin w \operatorname{rot}\boldsymbol{l}_2 - \operatorname{grad} w \times \boldsymbol{\beta},\\ \operatorname{rot}\boldsymbol{\beta} &= \sin w \operatorname{rot}\boldsymbol{l}_1 + \cos w \operatorname{rot}\boldsymbol{l}_2 + \operatorname{grad} w \times \boldsymbol{\nu},\end{aligned} \tag{22}$$

implied by (21). As a result, we obtain formula (17) for $\boldsymbol{R}_l^*$. From this, using (13) and (14) and the equality $k\boldsymbol{\beta} = \operatorname{rot}\boldsymbol{\tau}$, we obtain formulas (19) and (18), respectively. □

Note that equalities (18) and (19) are formally obtained from formulas (4) and (6), respectively, by replacing $\boldsymbol{R}^*\to\boldsymbol{R}_l^*$, $\varkappa\to\varkappa_l$, $\boldsymbol{\nu}\to\boldsymbol{l}_1$, and $\boldsymbol{\beta}\to\boldsymbol{l}_2$.

4. Conservation law for a family of surfaces

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then a family $\{S_\tau\}$ of surfaces S_τ in the domain D satisfies the divergent identity (conservation law)*

$$\operatorname{div}\{K\boldsymbol{\tau} + k_2(\boldsymbol{l}_2\cdot\operatorname{rot}\boldsymbol{\tau})\boldsymbol{l}_1 - k_1(\boldsymbol{l}_1\cdot\operatorname{rot}\boldsymbol{\tau})\boldsymbol{l}_2\} = 0 \tag{23}$$

$$\Leftrightarrow\ \operatorname{div}\{K\boldsymbol{\tau} + (H + B/2)\boldsymbol{K}_\tau - A\operatorname{rot}\boldsymbol{\tau}/2\} = 0 \tag{24}$$

$$\Leftrightarrow\ \operatorname{div}\{-\boldsymbol{\tau}\operatorname{div}\boldsymbol{S}_l^* + (\boldsymbol{l}_1\cdot\operatorname{rot}\boldsymbol{\tau})\operatorname{rot}\boldsymbol{l}_1 + (\boldsymbol{l}_2\cdot\operatorname{rot}\boldsymbol{\tau})\operatorname{rot}\boldsymbol{l}_2 + \varkappa_l\operatorname{rot}\boldsymbol{\tau}\} = 0. \tag{25}$$

Here the expression in braces { } is everywhere equal to $-\operatorname{rot}\boldsymbol{R}_l^* = -\operatorname{rot}\boldsymbol{R}^*$; $K = -\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{R}_l^*$;

$$\operatorname{div}\boldsymbol{S}_l^* = -K + (\boldsymbol{l}_1\cdot\operatorname{rot}\boldsymbol{\tau})(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{l}_1) + (\boldsymbol{l}_2\cdot\operatorname{rot}\boldsymbol{\tau})(\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{l}_2). \tag{26}$$

Proof. Formula (14) of the form

$$\operatorname{rot} \boldsymbol{R}^* = \frac{1}{2} \boldsymbol{\tau} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) - k\boldsymbol{\nu}(\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta}) - k\boldsymbol{\beta}(\varkappa + \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta}) \tag{27}$$

was obtained in [1]. Using the equalities $\boldsymbol{\nu} = \boldsymbol{l}_1 \cos w - \boldsymbol{l}_2 \sin w$ and $\boldsymbol{\beta} = \boldsymbol{l}_1 \sin w + \boldsymbol{l}_2 \cos w$, implied by (8), formulas (22), equality (9), and the well-known formula $\varkappa = -\frac{1}{2}(\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\nu} + \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta})$ [9], we obtain

$$\begin{aligned} \boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta} &= (\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_2) \cos^2 w - (\boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_1) \sin^2 w, \\ \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta} + \varkappa &= (\boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_1 + \boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_2) \sin w \cos w. \end{aligned}$$

From this it follows that $k\boldsymbol{\nu}(\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta}) + k\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta} + \varkappa) = k(\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_2)\boldsymbol{l}_1 \cos w + k(\boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_1)\boldsymbol{l}_2 \sin w$. Next we use equalities (11), the formulas $k \sin w = \boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{\tau}$ and $k \cos w = \boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{\tau}$ implied by (8) in view of the equalities $\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$ and $\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\tau} = k$, and the formula $K = -\frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$ for the Gaussian curvature K of the surface S_τ [9, Ch. 1, § 8]. As a result, from formula (27), we obtain the identity $\operatorname{rot} \boldsymbol{R}_l^* = \operatorname{rot} \boldsymbol{R}^* = -\{K\boldsymbol{\tau} + k_2(\boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{\tau})\boldsymbol{l}_1 - k_1(\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{\tau})\boldsymbol{l}_2\}$, which implies the conservation law (23).

Using the formulas $k\boldsymbol{\nu} = \boldsymbol{K}_\tau$, $k\boldsymbol{\beta} = \operatorname{rot} \boldsymbol{\tau}$, $\operatorname{div} \boldsymbol{\tau} = \operatorname{div}(\boldsymbol{\nu} \times \boldsymbol{\beta}) = \operatorname{rot} \boldsymbol{\nu} \cdot \boldsymbol{\beta} - \boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta}$, $\operatorname{div} \boldsymbol{\tau} = -2H$ [9, Ch. 1, § 5], and $\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta} = H + B/2$, and $\varkappa + \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta} = -A/2$, from (27) we have the identity $\operatorname{rot} \boldsymbol{R}_l^* = \operatorname{rot} \boldsymbol{R}^* = -\{K\boldsymbol{\tau} + (H + B/2)\boldsymbol{K}_\tau - A \operatorname{rot} \boldsymbol{\tau}/2\}$, which leads to the conservation law in the form of (24).

Formula (26) follows from (15) and the equalities $\operatorname{div} \boldsymbol{S}_l^* = \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) + \operatorname{rot} \boldsymbol{\tau} \cdot \boldsymbol{R}_l^* - \operatorname{rot} \boldsymbol{R}_l^* \cdot \boldsymbol{\tau}$, $\operatorname{rot} \boldsymbol{R}_l^* \cdot \boldsymbol{\tau} = \operatorname{rot} \boldsymbol{R}^* \cdot \boldsymbol{\tau} = \frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = -K$, and $\operatorname{rot} \boldsymbol{\tau} \cdot \boldsymbol{R}_l^* = (\operatorname{rot} \boldsymbol{l}_1 \cdot \boldsymbol{\tau})(\operatorname{rot} \boldsymbol{\tau} \cdot \boldsymbol{l}_1) + (\operatorname{rot} \boldsymbol{l}_2 \cdot \boldsymbol{\tau})(\operatorname{rot} \boldsymbol{\tau} \cdot \boldsymbol{l}_2)$ (in view of (19)). Expressing the quantity K from (26) and substituting it into (23), with the use of equalities (9) and (11), we obtain the conservation law in the form of (25). □

Remark 2. As shown in Section 3.3 in [1], the vector field $\boldsymbol{S}(\boldsymbol{\tau})$, as well as the fields $\boldsymbol{S}^*$ and $\boldsymbol{S}_l^*$, is the sum of three curvature vectors of some three curves mutually orthogonal at each point of the domain D.

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