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# About one inverse initial-boundary value problem for nonlinear one-dimensional poroelasticity equations

Kh.Kh. Imomnazarov, Sh.Kh. Imomnazarov, P.V. Korobov, A.E. Kholmuradov

**Abstract.** We consider a one-dimensional inverse boundary value problem for a nonlinear system of the poroelasticity equations. We obtain estimates for the conditional stability of the inverse problem.

**Keywords:** hyperbolic system, direct problem, Volterra's equation, porous medium, friction coefficient.

Let us consider the following one-dimensional initial boundary value problem for the nonlinear system of equations of poroelasticity

$$\begin{aligned} \rho_s u_{tt} &= (\mu(u_x)u_x)_x - \rho_l^2((u-v)\chi(u-v))_t, \\ \rho_l v_t &= \rho_l^2(u-v)\chi(u-v), \quad x \in (0,L), \quad t \in (0,T), \end{aligned} \tag{1}$$

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \quad v|_{t=0} = 0, \quad x \in (0,L), \tag{2}$$

$$\mu(u_x)u_x|_{x=L} = f(t), \quad u|_{x=0} = 0, \quad t \in (0,T). \tag{3}$$

Here $u$ and $v$ are the velocities of elastic porous body with a constant partial density $\rho_s = \rho_s^f(1-d_0)$ and of the fluid with a constant partial density $\rho_l = \rho_l^f d_0$, respectively, $d_0$ is porosity, $u_t = \dfrac{\partial u}{\partial t}$, $f : [0,T] \to R$, $u_0 : [0,L] \to R$, $u_1 : [0,L] \to R$, $\rho_s^f$ and $\rho_l^f$ are the physical density of elastic porous body and the fluid, respectively, $\mu(\nu)$ is a three times continuously differentiable positive function, $\chi(\nu)$ is a two times continuously differentiable positive function.

In this paper, using the ideas from [4], we study the inverse problem for the one-dimensional dynamical system of equations of porous media. The direct problem is considered in [1].

**The statement of the problem and formulation of results.** The problem of definition of $u$ and $v$ from (1)–(3) with given $\mu$, $\chi$, $\rho_s$, $\rho_l$ will be called a one-dimensional direct dynamic problem for porous media [1]. The inverse problem is to determine $u$, $v$, $\mu$ from (1)–(3) (with given $\chi$, $\rho_s$, $\rho_l$) with additional information $\tilde{u} := u(L,\cdot)$.

We introduce the functions $\tilde{\mu}(s) = s\mu(s)$, $\tilde{\chi}(s) = s\chi(s)$. To study the properties of our mathematical model, we consider the operator $F$, that is, mapping the function $\tilde{\mu}$ onto the given $\tilde{u} := u(L, \cdot)$, which is a restriction of the solution $u$ for the following initial boundary value problem

$$\begin{aligned} \rho_s u_{tt} &= (\tilde{\mu}(u_x))_x - \rho_l^2(\tilde{\chi}(u - v))_t, \\ v_t &= \rho_l \tilde{\chi}(u - v), \quad x \in (0, L), \quad t \in (0, T), \end{aligned} \tag{4}$$

with the initial conditions

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \quad v|_{t=0} = 0, \quad x \in (0, L), \tag{5}$$

and the boundary conditions

$$\tilde{\mu}(u_x)|_{x=L} = f(t), \quad u|_{x=0} = 0, \quad t \in (0, T). \tag{6}$$

Then the function $\tilde{\mu}$ can be found from the solution of the operator equation

$$F(\tilde{\mu}) = \tilde{u}. \tag{7}$$

The derivative of the operator $F$ in some direction $\delta\tilde{\mu}$ is calculated in the following way

$$F'(\tilde{\mu})[\delta\tilde{\mu}] = \widehat{u}(L, \cdot), \tag{8}$$

where the functions $\widehat{u}$, $\widehat{v}$ are the solution of the initial-boundary value problem

$$\begin{aligned} \rho_s \widehat{u}_{tt} &= (\tilde{\mu}'(u_x)\widehat{u}_x)_x - \rho_l^2(\tilde{\chi}'(u - v)(\widehat{u} - \widehat{v}))_t + (\delta\tilde{\mu}(u_x))_x, \\ \widehat{v}_t &= \rho_l \tilde{\chi}'(u - v)(\widehat{u} - \widehat{v}), \quad x \in (0, L), \quad t \in (0, T), \end{aligned} \tag{9}$$

with the initial conditions

$$\widehat{u}|_{t=0} = 0, \quad \widehat{u}_t|_{t=0} = 0, \quad \widehat{v}|_{t=0} = 0, \quad x \in (0, L), \tag{10}$$

and the boundary conditions

$$\widehat{u}|_{x=0} = 0, \quad \tilde{\mu}'(u_x)\widehat{u}_x + \delta\tilde{\mu}(u_x)|_{x=L} = 0, \quad t \in (0, T). \tag{11}$$

In formulas (9)–(11) the functions $u$, $v$ are the solution of the initial-boundary value problem (4)–(6).

Suppose that the following conditions are valid

$$u_0 \in C^3(0, L), \quad u_1 \in C^2(0, L), \quad f \in C^2(0, T), \tag{12}$$

and the compatibility conditions

$$
\begin{gathered}
(\tilde{\mu}^{-1} \circ f)(0) = u_0'(L), \quad (\tilde{\mu}^{-1} \circ f)'(0) = u_1'(L), \\
\rho_s(\tilde{\mu}^{-1} \circ f)''(0) = (\tilde{\mu}(u_0'))''(L) - \rho_l^2[(u_1 - \rho_l\tilde{\chi}(u_0))\tilde{\chi}'(u_0)]'(L)
\end{gathered}
\tag{13}
$$

on the boundary and

$$
u_0(0) = u_1(0) = u_0''(0) = u_1''(0) = 0 \tag{14}
$$

on the left boundary.

Assume that the functions $\tilde{\mu}$, $\tilde{\chi}$ belong to the set

$$
D(F) = \left\{ \begin{array}{l} (\tilde{\mu}, \tilde{\chi}) \in X \mid \tilde{\mu}'(s) \geq \mu_0,\ \tilde{\mu}''(s) \leq C,\ \tilde{\chi}''(s) \leq C \\ \text{for any } s \in [0, S], \text{ and condition (13) is fulfilled} \end{array} \right\}, \tag{15}
$$

for some positive constants $\mu_0$, $C$. Further, we denote by $C$ a positive constant that is greater than the previous $C$,

$$
X = \big\{(\tilde{\mu}, \tilde{\chi}) \in C^3(0, S) \times C^2(0, S) \mid \tilde{\mu}(0) = 0,\ \tilde{\chi}(0) = 0\big\}, \tag{16}
$$

where $S > 0$.

Using different norms both in the preimage and in the image spaces, we obtain a stable solution in the interval $[0, \bar{s}] \subseteq [0, S]$, the parameter curve $s \mapsto \tilde{\mu}(s)$ can be uniquely determined from the given measurements.

A difference $F(\breve{\mu}) - F(\tilde{\mu})$, $\breve{\mu}, \tilde{\mu} \in D(F)$, can be written as the right-hand side value for $\widehat{u}$, $\widehat{v}$ for the following initial-boundary value problem

$$
\begin{aligned}
\rho_s \widehat{u}_{tt} &= (a\widehat{u}_x + \phi)_x - \rho_l^2(b(\widehat{u} - \widehat{v}))_t, \\
\widehat{v}_t &= \rho_l b(\widehat{u} - \widehat{v}), \quad x \in (0, L), \quad t \in (0, T),
\end{aligned}
\tag{17}
$$

with zero initial conditions and the boundary conditions

$$
a\widehat{u}_x + \phi|_{x=L} = 0, \quad \widehat{u}|_{x=0} = 0, \quad t \in (0, T), \tag{18}
$$

where

$$
\begin{aligned}
a(x,t) &= \int_0^1 \tilde{\mu}'(\breve{u}_x(x,t) + (u_x(x,t) - \breve{u}_x(x,t))\theta)\, d\theta, \\
b(x,t) &= \int_0^1 \tilde{\chi}'\big(\breve{u}_x(x,t) - \breve{v}_x(x,t) + \\
&\qquad (u_x(x,t) - v_x(x,t) - (\breve{u}_x(x,t) - \breve{v}_x(x,t))\theta\big)\, d\theta, \\
\phi(x,t) &= \delta\tilde{\mu}(u_x(x,t)), \quad \delta\tilde{\mu} = \breve{\mu} - \tilde{\mu}.
\end{aligned}
\tag{19}
$$

The functions $\breve{u}$, $\breve{v}$ are the solution of the initial-boundary value problem (4)–(6) with $\tilde{\mu} = \breve{\mu}$, i.e.

$$
F(\breve{\mu}) - F(\tilde{\mu}) = \hat{u}(L, \cdot). \tag{20}
$$

First, consider the initial boundary value problem (17)–(18) in the case of constant coefficients, i.e. $a(x,t) = \bar{a}$, $b(x,t) = \bar{b}$, $\bar{a}, \bar{b} \in R$. Therefore, we consider the following initial-boundary value problem

$$\begin{aligned} \rho_s \widehat{u}_{tt} &= \bar{a}\widehat{u}_{xx} - \rho_l^2 \bar{b}(\widehat{u}_t - \widehat{v}_t) + \phi_x, \\ \widehat{v}_t &= \rho_l \bar{b}(\widehat{u} - \widehat{v}), \quad x \in (0,L), \quad t \in (0,T), \end{aligned} \tag{21}$$

with zero initial conditions

$$\widehat{u}|_{t=0} = 0, \quad \widehat{u}_t|_{t=0} = 0, \quad \widehat{v}|_{t=0} = 0, \quad x \in (0,L), \tag{22}$$

and boundary conditions

$$\bar{a}\widehat{u}_x + \phi|_{x=L} = 0, \quad \widehat{u}|_{x=0} = 0, \quad t \in (0,T). \tag{23}$$

Using the method of characteristics, initial boundary value problem (21)–(23) is reduced to Volterra's equation of the first kind for the difference $\delta\tilde{\mu}$ between the parameters of curves.

**Theorem 1.** *Let the functions $\hat{u}$, $\hat{v}$ be the solution of the initial boundary value problem* (21)–(23)*. The function $\phi$ is defined by formula* (19) *for $u \in C^{3,2}([0,L] \times [0,T])$, $v \in C^{0,1}([0,L] \times [0,T])$, satisfying boundary conditions* (6) *and initial conditions* (5) *with condition of smoothness* (12)*, $f(0) = 0$ and $f$ is a strictly monotoniously increasing function, $u_0' \equiv 0$, $\tilde{\mu} \in D(F)$, and $\delta\tilde{\mu} \in C^2([0,S_1])$ for some $S_1 > 0$ such that*

$$\big\{u_x(x,t) \mid (x,t) \in [0,L] \times [0,T]\big\} \subseteq [0,S_1].$$

*Furthermore, assume that*

$$\left|\pm\sqrt{\frac{\bar{a}}{\rho_s}}\, u_{xx}(x,t) + u_{xt}(x,t)\right| \geq c_1 \quad \forall\, (x,t) \in (0,L) \times (0,\bar{t}) \tag{24}$$

*holds for some $c_1 > 0$, $0 < \bar{t} \leq T$.*

*Then, with*

$$\bar{s} = \tilde{\mu}^{-1}(f(\bar{t})) > 0 \tag{25}$$

*the estimate of $l$-stability* [4] *is valid*

$$\|\delta\tilde{\mu}\|_{L_2(0,\bar{s})} \leq C\Big\{\|\hat{u}(L,\cdot)\|_{H^1(0,\bar{t})} + \rho_l^3 \|\hat{u}\|_{H^1((0,\bar{t})\times(0,\bar{t}))}\Big\} \tag{26}$$

*with some constant $C > 0$.*

**Theorem 2.** *Let the conditions of Theorem 1 be fulfilled and*

$$f(0)=0, \quad f(t)\geq 0, \quad f'(t)\geq f_0>0 \quad \forall\, t\in[0,\bar{t}], \tag{27}$$

$$u_0'(x)=0 \quad \forall\, x\in[0,L] \tag{28}$$

*for some* $f_0$. *Let* $\tilde{\mu}\in D(F)$, *and* $u$, $v$ *be solutions of the initial-boundary value problem* (4)–(6).

*Additionally, assume that*

$$\left|\Big(\pm\sqrt{\frac{\tilde{\mu}'(u_x)}{\rho_s}}u_{xx}+u_{xt}\Big)(x(t),t)\right|\geq c_1 \quad \forall\, t\in[0,\bar{t}], \tag{29}$$

*performed on some segment* $[0,\bar{t}]\subseteq[0,T]$ *with some* $c_1>0$, *for all the characteristic curves* $t\mapsto x(t)$ *of* (4), *and* $\bar{t}$, $L$ *are small enough.*

*Then the function* $u(L,t)$, $t\in[0,\bar{t}]$, *uniquely determines* $\tilde{\mu}$ *on the interval* $[0,\bar{s}]$, *where*

$$\bar{s}=\tilde{\mu}^{-1}(f(\bar{t}))>0 \tag{30}$$

*and the estimate of* $l$*-stability is valid*

$$\|\breve{\mu}-\tilde{\mu}\|_{L_2(0,\bar{s})}\leq C\Big\{\|F(\breve{\mu})-F(\tilde{\mu})\|_{H^1(0,\bar{t})}+\rho_l^3\|\hat{u}\|_{H^1((0,\bar{t})\times(0,\bar{t}))}\Big\} \tag{31}$$

*with some constant* $C>0$ *for all* $\tilde{\mu}\in D(F)\cap B_r(\tilde{\mu})$, *where* $B_r(\tilde{\mu})$ *is a ball of sufficiently small radius* $r$ *(in* $C^3$ *norm) with the center* $\tilde{\mu}$.

**Proof of Theorems.** For simplicity, assume that $\rho_s=L=a=1$. For the sake of convenience exclude the function $\widehat{v}$ from the equation of motion for $\widehat{u}$. These functions satisfy the relations (17)–(20):

$$\begin{gathered}\widehat{u}_{tt}=\widehat{u}_{xx}-b\rho_l^2\widehat{u}_t+b^2\rho_l^3\widehat{u}-b^3\rho_l^4\int_0^t e^{-b\rho_l(t-\tau)}\widehat{u}(x,\tau)\,d\tau+\phi_x,\\ \widehat{u}_x(1,t)+\phi=0, \quad m(t):=\widehat{u}(1,t),\\ \widehat{v}(x,t)=b\rho_l\int_0^t e^{-b\rho_l(t-\tau)}\widehat{u}(x,\tau)\,d\tau.\end{gathered} \tag{32}$$

We represent $\widehat{u}=p\,e^{-b\rho_l^2 t/2}$. For the function $p$ we obtain the following problem

$$\begin{gathered}p_{tt}=p_{xx}+Ap-b^3\rho_l^4\int_0^t e^{-B(t-\tau)}p(x,\tau)d\tau+\tilde{\phi}_x,\\ p_x(1,t)+\tilde{\phi}=0, \qquad p(1,t)=\tilde{m}(t),\end{gathered} \tag{33}$$

where

$$A=b^2\rho_l^3\Big(1+\frac{\rho_l}{4}\Big), \quad B=b\rho_l\Big(1-\frac{\rho_l}{2}\Big), \quad \tilde{\phi}=\phi e^{b\rho_l^2 t/2}, \quad \tilde{m}=me^{b\rho_l^2 t/2}.$$

Solution of problem (33) has the form [3]

$$
\begin{aligned}
p(x,t) =& \frac{1}{2}\Big[\tilde{m}(1+t-x) + \tilde{m}(1+t-x-\min\{1+t-x,\, 2(1-x)\})\Big] + \\
& \frac{1}{2}\int_{1+t-x-\min\{1+t-x,2(1-x)\}}^{1+t-x} \tilde{\phi}(1,\eta)\,d\eta - \\
& \int_0^{\min\{\frac{1}{2}(1+t-x),1-x\}} \int_0^{\eta} \tilde{\phi}_x(1-\tau,\tau+1+t-x-2\eta)\,d\tau\,d\eta - \\
& \int_{\min\{\frac{1}{2}(1+t-x),1-x\}}^{1-x} \int_{2\eta-(1+t-x)}^{\eta} \tilde{\phi}_x(1-\tau,\tau+1+t-x-2\eta)\,d\tau\,d\eta - \\
& \int_0^{\min\{\frac{1}{2}(1+t-x),1-x\}} \int_0^{\eta} P(1-\tau,\tau+1+t-x-2\eta)\,d\tau\,d\eta - \\
& \int_{\min\{\frac{1}{2}(1+t-x),1-x\}}^{1-x} \int_{2\eta-(1+t-x)}^{\eta} P(1-\tau,\tau+1+t-x-2\eta)\,d\tau\,d\eta,
\end{aligned}
\tag{34}
$$

where

$$P = Ap - b^3\rho_l^4 \int_0^t e^{-B(t-\tau)} p(x,\tau)\,d\tau.$$

From the initial and boundary conditions at the left boundary for $\widehat{u}$ we obtain

$$p(x,0) = P(x,0) = 0, \qquad p(0,t) = P(0,t) = 0.$$

From (32), (33) it follows that $\tilde{m}(0) = m(0) = 0$.

Repeating the arguments from [4] relative to $\tilde{\phi}$, we obtain Volterra's integral equation of the first kind

$$
\begin{aligned}
-\tilde{m}(t) =& \int_0^t \tilde{\phi}(|\sigma-t+1|,\sigma)\,d\sigma - 2\int_0^{t/2}\int_0^{\eta} P(1-\tau,\tau+t-2\eta)\,d\tau\,d\eta - \\
& 2\int_{t/2}^{t}\int_{2\eta-t}^{\eta} P(1-\tau,\tau+t-2\eta)\,d\tau\,d\eta
\end{aligned}
\tag{35}
$$

In the first integral we make a change in the variables

$$\lambda := u_x(|\eta-t+1|,\eta), \quad \tau := f^{-1}(\tilde{\mu}(\lambda)).$$

Then we have

$$
\begin{aligned}
\int_0^t \tilde{\phi}(|\sigma-t+1|,\sigma)\,d\sigma &= \int_{u_x(|t-1|,0)}^{u_x(1,t)} k(\lambda,t)\delta\mu(\lambda)\,d\lambda = \int_0^{\tilde{\mu}^{-1}(f(t))} k(\lambda,t)\delta\mu(\lambda)\,d\lambda \\
&= \int_0^t k(\tilde{\mu}^{-1}(f(\tau)),t)\frac{f'(\tau)}{\tilde{\mu}'(\tilde{\mu}^{-1}(f(\tau)))}\delta\mu(\tilde{\mu}^{-1}(f(\tau)))\,d\tau \quad \forall\, t\in[0,\bar{t}],
\end{aligned}
$$

where

$$k(\lambda, t) = \frac{e^{b\rho_l^2 \eta/2}}{\operatorname{sgn}(\eta - t + 1) u_{xx}(|\eta - t + 1|, \eta) + u_{xt}(|\eta - t + 1|, \eta)},$$

$\eta = \eta(\lambda, t - 1)$ according to the theorem of the implicit function.

Supplying this ratio in (35) relative to $\delta\mu \circ \tilde{\mu}^{-1} \circ f$ we obtain Volterra's integral equations of the first kind

$$\begin{aligned} -m(t) e^{b\rho_l^2 t/2} = \int_0^t k(\tilde{\mu}^{-1}(f(\tau)), t) \frac{f'(\tau)}{\tilde{\mu}'(\tilde{\mu}^{-1}(f(\tau)))} \delta\mu(\tilde{\mu}^{-1}(f(\tau)))\, d\tau - \\ 2 \int_0^{t/2} \int_0^{\eta} P(1 - \tau, \tau + t - 2\eta)\, d\tau\, d\eta - \\ 2 \int_{t/2}^{t} \int_{2\eta - t}^{\eta} P(1 - \tau, \tau + t - 2\eta)\, d\tau\, d\eta \quad \forall\, t \in [0, \bar{t}]. \end{aligned} \tag{36}$$

Note that the kernel $k(\tilde{\mu}^{-1}(f(\tau)), t) \frac{f'(\tau)}{\tilde{\mu}'(\tilde{\mu}^{-1}(f(\tau)))}$ is limited, differentiable with respect to $t$ separated from zero diagonal $\tau = t$. According to the theory of Volterra's integral operators [4, 7], from (36) we obtain

$$\begin{aligned} \|\delta\mu\|_{L_2(0,\bar{\lambda})} &\le \frac{\|f\|_{C^1}}{\mu_0} \|\delta\mu \circ \tilde{\mu}^{-1} \circ f\|_{L_2(0,\bar{t})} + C\rho_l^3 \|\widehat{u}\|_{H^1((0,\bar{t})\times(0,\bar{t}))} \\ &\le C\Big\{ \|\tilde{m}'\|_{L_2(0,\bar{t})} + \rho_l^3 \|\widehat{u}\|_{H^1((0,\bar{t})\times(0,\bar{t}))} \Big\}. \end{aligned}$$

Hence, taking into account the definitions of $\tilde{m}$, we obtain estimate (26). Theorem 1 is proved.

The proof of Theorem 2 is carried out in the same manner as in [4], using Theorem 1. □

## References


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