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Conservation laws for the two-velocity hydrodynamics equations with one pressure*

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Abstract. A series of the differential identities connecting velocities, pressure and body force in the two-velocity hydrodynamics equations with equilibrium of pressure phases are found. Some of these identities have a divergent form and can be considered to be certain conservation laws. It is detected that the flow functions for the plane motion satisfy the Monge–Ampere system of equations.

Keywords. Two-velocity hydrodynamics, hyperbolic system.

1. Introduction

In the vector analysis, in the theory of field and in the mathematical physics, an important role is played by differential identities of a classical kind. In paper [1], the generalization of some identities of the theory of inverse problems for the kinetic equations is obtained. In paper [2], a set of formulas of the vector analysis in the form of differential identities of second and third orders connecting the Laplacian of an arbitrary smooth scalar function $u(x, y)$ of two independent variables, the module of its gradient, the angular value and the direction of its gradient have been obtained. Representation of the Gaussian curvature of a surface in the three-dimensional Euclidean space with a graph $z = u(x, y)$ is found. Some of its generalizations and similar formulas for a surface in a pseudo-Euclidean space are given. The results of paper [2] are generalized in [3] in the two directions: a three-dimensional case and any (not necessarily potential) smooth vector field $\boldsymbol{v}$. A set of formulas of the vector analysis in the form of differential identities which, on the one hand, connect the module $|\boldsymbol{v}|$ and the direction $\boldsymbol{\tau}$ of any smooth vector field $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$ in the three-dimensional ($\boldsymbol{v} = \boldsymbol{v}(x, y, z)$) and in the two-dimensional ($\boldsymbol{v} = \boldsymbol{v}(x, y)$) cases, are obtained. On the other hand, the formulas are found separately in the sense of the module $|\boldsymbol{v}|$ and the direction $\boldsymbol{\tau}$ of the vector field $\boldsymbol{v} = |\boldsymbol{v}|\boldsymbol{\tau}$. Namely, the basic identity to any smooth vector field $\boldsymbol{v}$ explicitly compares a vector field $\boldsymbol{Q} = \boldsymbol{P} + \boldsymbol{S}$, where $\boldsymbol{P}$ is defined only by the module $|\boldsymbol{v}|$ of the field $\boldsymbol{v}$ and is potential both in a two-dimensional, and in a three-dimensional cases, and the field $\boldsymbol{S}$ is defined only by the direction $\boldsymbol{\tau}$ of the field $\boldsymbol{v}$ and is solenoidal in a

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two-dimensional case. Applications of the identities obtained to the Euler hydrodynamic equation are presented.

In this paper, applications of the obtained identities [3] to the two-velocity hydrodynamics equations with one pressure are given.

2. A.G. Megrabov's differential identities connecting the modulus and the direction of a vector field

In paper [3], the following theorem was obtained:

Theorem 1. *For any vector field* $\boldsymbol{v} = \boldsymbol{v}(x, y, z) = |\boldsymbol{v}|\boldsymbol{\tau}$ *with components* $v_k(x, y, z) \in C^1(D)$, $k = 1, 2, 3$, *the module* $|\boldsymbol{v}| \neq 0$ *in* D *and the direction* $\boldsymbol{\tau}$, *the following identity is valid*

$$\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{v}) = \boldsymbol{P}(|\boldsymbol{v}|) + \boldsymbol{S}(\boldsymbol{\tau}), \tag{1}$$

where

$$\boldsymbol{Q}(\boldsymbol{v}) \overset{\text{def}}{=} \frac{\boldsymbol{v} \operatorname{div} \boldsymbol{v} + \boldsymbol{v} \times \operatorname{rot} \boldsymbol{v}}{|\boldsymbol{v}|^2}, \qquad \boldsymbol{P}(|\boldsymbol{v}|) \overset{\text{def}}{=} \nabla \ln |\boldsymbol{v}| = \frac{\nabla |\boldsymbol{v}|^2}{|\boldsymbol{v}|^2}, \tag{2}$$

$$\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{\tau}) \overset{\text{def}}{=} \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} + \boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau} = \boldsymbol{Q}(\boldsymbol{v}) - \boldsymbol{P}(|\boldsymbol{v}|). \tag{3}$$

For the vector field $\boldsymbol{S}$, *any of the following representations holds:*

$$\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} - \boldsymbol{\tau}_s = -\{(\boldsymbol{\tau} \times \nabla) \times \boldsymbol{\tau} + (\boldsymbol{\tau} \cdot \nabla)\, \boldsymbol{\tau}\} = -\frac{(\boldsymbol{v} \times \nabla) \times \boldsymbol{v}}{|\boldsymbol{v}|^2} \tag{4}$$

($\boldsymbol{\tau}_s = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau}$ *is a derivative of the vector* $\boldsymbol{\tau}$ *in the direction* $\boldsymbol{\tau}$),

$$\boldsymbol{S} = \operatorname{rot}(\alpha \boldsymbol{k}) - \cos^2 \theta \operatorname{rot}(\alpha \boldsymbol{k} - \tan \theta\, \boldsymbol{\lambda}) = \operatorname{rot}(\alpha \boldsymbol{k} + \cos \theta\, \boldsymbol{\psi}) - 2 \cos \theta \operatorname{rot} \boldsymbol{\psi}, \tag{5}$$

where $\boldsymbol{\lambda} = -\sin \alpha\, \boldsymbol{i} + \cos \alpha\, \boldsymbol{j}$, $\boldsymbol{\psi} = -\sin \theta\, \boldsymbol{\lambda} + \alpha \cos \theta\, \boldsymbol{k}$,

$$\boldsymbol{S} = -\nabla \alpha \times (\cos \theta\, \boldsymbol{\tau} - \boldsymbol{k}) + \nabla \theta \times \boldsymbol{\lambda}, \qquad \boldsymbol{S} = \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} - \kappa \boldsymbol{\nu}. \tag{6}$$

Here κ *is the curvature of the vector line of the field* $\boldsymbol{v}$ *and* $\boldsymbol{\nu}$ *is its unit normal. The following formula is valid:* $\kappa^2 = \sin^2 \theta\, \alpha_s^2 + \theta_s^2$, *where* $\alpha_s = (\nabla \alpha \cdot \boldsymbol{\tau})$ *and* $\theta_s = (\nabla \theta \cdot \boldsymbol{\tau})$ *are derivatives of the angles* α *and* θ *in the direction* $\boldsymbol{\tau}$, *respectively.*

The basic identity (1) *can also be presented in any of the forms*:

$$\boldsymbol{Q} + \boldsymbol{H}_i = \nabla \ln |\boldsymbol{v}| + \operatorname{rot} \boldsymbol{F}_i, \quad i = 1, 2,$$

where $\boldsymbol{H}_1 = \cos^2 \theta \operatorname{rot}(\alpha \boldsymbol{k} - \tan \theta\, \boldsymbol{\lambda})$, $\boldsymbol{H}_2 = 2 \cos \theta \operatorname{rot} \boldsymbol{\psi}$, $\boldsymbol{F}_1 = \alpha \boldsymbol{k}$, $\boldsymbol{F}_2 = \alpha \boldsymbol{k} + \cos \theta\, \boldsymbol{\psi}$, *thus the vectors* $\boldsymbol{H}_i$, $\boldsymbol{F}_i$, *as well as* $\boldsymbol{S}$, *are defined only by the angles* α, θ, *i.e. the direction* $\boldsymbol{\tau}$ *of the field* $\boldsymbol{v}$.

If the presence of the property $|\boldsymbol{v}| \neq 0$ *in* D *is not assumed, then* (1) *takes the form*

$$\boldsymbol{W} = \boldsymbol{v}\operatorname{div}\boldsymbol{v} + \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v} = \nabla|\boldsymbol{v}|^2 - \boldsymbol{V},$$

where

$$\boldsymbol{V} \stackrel{\text{def}}{=} -|\boldsymbol{v}|^2\boldsymbol{S} = \frac{1}{2}\nabla|\boldsymbol{v}|^2 - \boldsymbol{v}\operatorname{div}\boldsymbol{v} - \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}$$
$$= -|\boldsymbol{v}|^2\{\boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau}\} = \boldsymbol{v}\times\nabla\times\boldsymbol{v}.$$

Other formulas for $\boldsymbol{W}$ *and* $\boldsymbol{v}$ *are obtained by substituting any expression for* $\boldsymbol{S}$ *from* (4)–(6) *in the latter equality.*

Theorem 2. *Under condition of Theorem 1 and provided* $v_k(x,y,z) \in C^2(D)$, $k = 1,2,3$, *the following formulas are valid*:

$$\operatorname{div}\boldsymbol{S} = -2\sin\theta\,(\boldsymbol{\tau}\cdot\boldsymbol{B}) = -\frac{2\sin\theta\,(\boldsymbol{v}\cdot\boldsymbol{B})}{|\boldsymbol{v}|},$$

where $\boldsymbol{B} = \nabla\alpha\times\nabla\theta = \operatorname{rot}(\alpha\nabla\theta) = -\operatorname{rot}(\theta\nabla\alpha)$. *In addition, the following identity takes place*

$$\operatorname{div}(\boldsymbol{Q} - \boldsymbol{P} + \boldsymbol{H}_i) = 0 \quad \Longleftrightarrow$$
$$\operatorname{div}\left\{\frac{\boldsymbol{v}\operatorname{div}\boldsymbol{v} + \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}}{|\boldsymbol{v}|^2} - \nabla\ln|\boldsymbol{v}| + \boldsymbol{H}_i\right\} = 0, \quad i = 1,2,$$

which can be considered to be a conservation law (*its differential form*) *with the integrated form for a flux* $\int_S([\boldsymbol{Q} - \boldsymbol{P} + \boldsymbol{H}_i]\cdot\boldsymbol{\eta})\,dS = 0$, *where* S *is a piecewise smooth boundary of the domain* D *with normal* $\boldsymbol{\eta}$.

In Theorems 1 and 2, the following notations are used: symbols $(\boldsymbol{a}\cdot\boldsymbol{b})$ and $(\boldsymbol{a}\times\boldsymbol{b})$ denote the scalar and the vector products of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively; ∇ is the Hamiltonian operator (a nabla); Δ is the Laplace operator; D is a domain in the space x, y, z; $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are unit vectors on the axes x, y, z; $\boldsymbol{v} = \boldsymbol{v}(x,y,z) = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ is a vector field defined in D, $v_k = v_k(x,y,z)$ are scalar functions, $k = 1,2,3$, $|\boldsymbol{v}|^2 = v_1^2 + v_2^2 + v_3^2$; $\alpha = \alpha(x,y,z)$ is the angle of the slope of the vector $v_1\boldsymbol{i} + v_2\boldsymbol{j}$ to the axis Ox, so that $\cos\alpha = v_1/\sqrt{v_1^2 + v_2^2}$, $\sin\alpha = v_2/\sqrt{v_1^2 + v_2^2}$, i.e. $\alpha(x,y,z)$ is the polar angle of a point $(\xi = v_1,\ \varsigma = v_2)$ on the plane ξ, ς or the argument $\operatorname{Arg} w$ of a complex number $w = \xi + i\,\varsigma$ (i is an imaginary unit):

$$\alpha \stackrel{\text{def}}{=} \arctan\frac{v_2}{v_1} + (2k + \delta)\pi, \quad k \in \mathbb{Z}, \tag{7}$$

$\delta = 0$ and $\delta = 1$ in quadrants I, IV and II, III of the plane ξ, ς, respectively; $\theta = \theta(x,y,z)$ is the angle between the vector $\boldsymbol{v}$ and the axis Oz: $\theta \stackrel{\text{def}}{=}$

$\arccos \frac{v_3}{|\boldsymbol{v}|}$, so that $0 \le \theta \le \pi$, $\cos\theta = \frac{v_3}{|\boldsymbol{v}|}$, $\sin\theta = \frac{\sqrt{v_1^2+v_2^2}}{|\boldsymbol{v}|}$. This means that α, θ are spherical coordinates on the space $\xi = v_1$, $\varsigma = v_2$, $\zeta = v_3$. Thus, $\boldsymbol{v} = |\boldsymbol{v}|\,\boldsymbol{\tau}$, where $\boldsymbol{\tau} = \boldsymbol{\tau}(\alpha,\theta) = \cos\alpha\sin\theta\,\boldsymbol{i} + \sin\alpha\sin\theta\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}$ is the direction of the vector field $\boldsymbol{v}$ ($|\boldsymbol{\tau}| = 1$).

In a two-dimensional case, $\boldsymbol{v} = \boldsymbol{v}(x,y) = v_1\,\boldsymbol{i} + v_2\,\boldsymbol{j} = \boldsymbol{v}\,|\boldsymbol{\tau}|$, $v_3 \equiv 0$, $\theta \equiv \pi/2 \Rightarrow \boldsymbol{\tau} = \boldsymbol{\tau}(\alpha) = \cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{j}$, the angle α being defined by formula (7), $\nabla\theta = \boldsymbol{B} = 0$; $\forall\,\varphi(x,y) \in C^1(D)$, we have $\operatorname{rot}(\varphi\,\boldsymbol{k}) = \varphi_y\boldsymbol{i} - \varphi_x\boldsymbol{j}$, where $\varphi_x = \frac{\partial\varphi}{\partial x}$, $\varphi_y = \frac{\partial\varphi}{\partial y}$.

From Theorem 1 follows

Theorem 3. *For any plane vector field $\boldsymbol{v}(x,y)$ with the components $v_k(x,y) \in C^1(D)$, $k = 1,2$, the modulus $|\boldsymbol{v}| \ne 0$ in D and the direction $\boldsymbol{\tau} = \boldsymbol{\tau}(\alpha)$, we have the identity*

$$\boldsymbol{Q} \stackrel{\text{def}}{=} \frac{\boldsymbol{v}\operatorname{div}\boldsymbol{v} + \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}}{|\boldsymbol{v}|^2} = \nabla\ln|\boldsymbol{v}| + \operatorname{rot}(\alpha\,\boldsymbol{k}) \quad\Longrightarrow$$
$$\operatorname{div}\boldsymbol{v} = (\{\nabla\ln|\boldsymbol{v}| + \operatorname{rot}(\alpha\,\boldsymbol{k})\}\cdot\boldsymbol{v}), \quad \operatorname{rot}\boldsymbol{v} = \{\nabla\ln|\boldsymbol{v}| + \operatorname{rot}(\alpha\,\boldsymbol{k})\}\times\boldsymbol{v}. \quad (8)$$

Thus, $\boldsymbol{S} = \operatorname{rot}(\alpha\,\boldsymbol{k}) \Rightarrow (\boldsymbol{S}\cdot\nabla\alpha) = 0$, i.e., the vector lines of the vector field $\boldsymbol{S}$ coincide with the lines of the level of the scalar field of the angles $\alpha(x,y)$. If $v_k(x,y) \in C^2(D)$, $k = 1,2$, the following identities are valid:

$$\begin{aligned}
&\operatorname{div}\boldsymbol{S} = 0, \quad \operatorname{rot}\boldsymbol{S} = -(\Delta\alpha)\boldsymbol{k} \quad\Longrightarrow\\
&\Delta\ln|\boldsymbol{v}| = \operatorname{div}\boldsymbol{Q}, \quad (\Delta\alpha)\boldsymbol{k} = -\operatorname{rot}\boldsymbol{Q} \quad\Longrightarrow\\
&\Delta\operatorname{Ln}\{|\boldsymbol{v}|e^{\pm i\alpha}\} = \operatorname{div}\boldsymbol{Q} \mp i(\operatorname{rot}\boldsymbol{Q}\cdot\boldsymbol{k}).
\end{aligned}$$

In the conservation law of Theorem 2 we have $\boldsymbol{H}_i = 0$.

As is known [4], any smooth vector field can be presented in the form of the sum of a gradient of some scalar and a rotor of a certain vector. Identity (8) gives such a representation for the vector field $\boldsymbol{Q}$. At $\boldsymbol{v} = \nabla u(x,y)$, Theorem 3 gives the identity from paper [2].

3. Two-velocity hydrodynamics equations with one pressure

In papers [5, 6], based on conservation laws, the invariance of the equations concerning the Galilee transformation and conditions of thermodynamic conditioning, a nonlinear two-velocity model of motion of a liquid through deformable porous media is constructed. The two-velocity hydrodynamic theory with a condition of balance of pressure phases, has been constructed in paper [7]. The equation of motion of the two-velocity media with one pressure in the system in an isothermal case looks like [7]:

$$\frac{\partial\bar\rho}{\partial t}+\operatorname{div}(\tilde\rho\tilde{\boldsymbol v}+\rho\boldsymbol v)=0,\quad \frac{\partial\tilde\rho}{\partial t}+\operatorname{div}(\tilde\rho\tilde{\boldsymbol v})=0, \tag{9}$$

$$\frac{\partial\boldsymbol v}{\partial t}+(\boldsymbol v,\nabla)\boldsymbol v=-\frac{\nabla p}{\bar\rho}+\frac{\tilde\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol v}-\boldsymbol v)^2+\boldsymbol f, \tag{10}$$

$$\frac{\partial\tilde{\boldsymbol v}}{\partial t}+(\tilde{\boldsymbol v},\nabla)\tilde{\boldsymbol v}=-\frac{\nabla p}{\bar\rho}-\frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol v}-\boldsymbol v)^2+\boldsymbol f, \tag{11}$$

where $\tilde{\boldsymbol v}$ and $\boldsymbol v$ are the vectors of velocities of the subsystems making up a two-velocity continuum with the corresponding partial densities $\tilde\rho$ and ρ, $\bar\rho=\tilde\rho+\rho$ is the general density of the continuum; $p=p(\bar\rho,(\tilde{\boldsymbol v}-\boldsymbol v)^2)$ is the equation of state of the continuum; $\boldsymbol f$ is the vector of the mass force carried to the mass unit.

In terms of the vectors $\boldsymbol W$, $\boldsymbol V$, $\boldsymbol S$, $\boldsymbol Q$, $\boldsymbol P$, $\boldsymbol H_i$, $\boldsymbol F_i$, $\tilde{\boldsymbol W}$, $\tilde{\boldsymbol V}$, $\tilde{\boldsymbol S}$, $\tilde{\boldsymbol Q}$, $\tilde{\boldsymbol P}$, $\tilde{\boldsymbol H}_i$, $\tilde{\boldsymbol F}_i$, determined in Theorem 1, the system of equations (10), (11) can be written down in any of the following forms (symbols without tilde and with a tilde concern the corresponding subsystems of the continuum):

$$\begin{aligned}\boldsymbol W&=\frac{\partial\boldsymbol v}{\partial t}+\boldsymbol v\operatorname{div}\boldsymbol v+\frac12\nabla v^2+\frac{\nabla p}{\bar\rho}-\frac{\tilde\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol v}-\boldsymbol v)^2-\boldsymbol f,\\ -\boldsymbol V&=\frac{\partial\boldsymbol v}{\partial t}+\boldsymbol v\operatorname{div}\boldsymbol v+\frac{\nabla p}{\bar\rho}-\frac{\tilde\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol v}-\boldsymbol v)^2-\boldsymbol f,\end{aligned} \tag{12}$$

$$\boldsymbol G\overset{\text{def}}{=}\frac{1}{v^2}\left\{\frac{\partial\boldsymbol v}{\partial t}+\boldsymbol v\operatorname{div}\boldsymbol v+\frac{\nabla p}{\bar\rho}-\frac{\tilde\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol v}-\boldsymbol v)^2-\boldsymbol f\right\}=\boldsymbol S\iff$$

$$\boldsymbol G+\boldsymbol H_i=\operatorname{rot}\boldsymbol F_i,\quad i=1,2; \tag{13}$$

$$\begin{aligned}\tilde{\boldsymbol W}&=\frac{\partial\tilde{\boldsymbol v}}{\partial t}+\tilde{\boldsymbol v}\operatorname{div}\tilde{\boldsymbol v}+\frac12\nabla\tilde v^2+\frac{\nabla p}{\bar\rho}+\frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol v}-\boldsymbol v)^2-\boldsymbol f,\\ -\tilde{\boldsymbol V}&=\frac{\partial\tilde{\boldsymbol v}}{\partial t}+\tilde{\boldsymbol v}\operatorname{div}\tilde{\boldsymbol v}+\frac{\nabla p}{\bar\rho}+\frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol v}-\boldsymbol v)^2-\boldsymbol f,\end{aligned} \tag{14}$$

$$\tilde{\boldsymbol G}\overset{\text{def}}{=}\frac{1}{\tilde v^2}\left\{\frac{\partial\tilde{\boldsymbol v}}{\partial t}+\tilde{\boldsymbol v}\operatorname{div}\tilde{\boldsymbol v}+\frac{\nabla p}{\bar\rho}+\frac{\rho}{2\bar\rho}\nabla(\tilde{\boldsymbol v}-\boldsymbol v)^2-\boldsymbol f\right\}=\tilde{\boldsymbol S}\iff$$

$$\tilde{\boldsymbol G}+\tilde{\boldsymbol H}_i=\operatorname{rot}\tilde{\boldsymbol F}_i,\quad i=1,2. \tag{15}$$

In the case of a homogeneous incompressible medium, i.e., provided that $\rho=\text{const}$, $\tilde\rho=\text{const}\Rightarrow\operatorname{div}\boldsymbol v=0$, $\operatorname{div}\tilde{\boldsymbol v}=0\Leftrightarrow\boldsymbol v=\operatorname{rot}\boldsymbol A$, $\tilde{\boldsymbol v}=\operatorname{rot}\tilde{\boldsymbol A}$, where $\boldsymbol A$ and $\tilde{\boldsymbol A}$ are the vector potentials of the velocities $\boldsymbol v$ and $\tilde{\boldsymbol v}$, respectively, the two-velocity hydrodynamics equations are represented in the form

$$\begin{aligned}\boldsymbol W&=\nabla\left\{\frac12 v^2+\frac{p}{\bar\rho}+U-\frac{\tilde\rho}{2\bar\rho}(\tilde{\boldsymbol v}-\boldsymbol v)^2\right\}+\operatorname{rot}\{\boldsymbol A_t+\boldsymbol M\},\\ -\boldsymbol V&=\nabla\left\{\frac{p}{\bar\rho}+U-\frac{\tilde\rho}{2\bar\rho}(\tilde{\boldsymbol v}-\boldsymbol v)^2\right\}+\operatorname{rot}\{\boldsymbol A_t+\boldsymbol M\},\\ \tilde{\boldsymbol W}&=\nabla\left\{\frac12\tilde v^2+\frac{p}{\bar\rho}+U+\frac{\rho}{2\bar\rho}(\tilde{\boldsymbol v}-\boldsymbol v)^2\right\}+\operatorname{rot}\{\tilde{\boldsymbol A}_t+\boldsymbol M\},\end{aligned}$$

$$-\tilde{\boldsymbol{V}} = \nabla\left\{\frac{p}{\bar{\rho}} + U + \frac{\rho}{2\bar{\rho}}(\tilde{\boldsymbol{v}} - \boldsymbol{v})^2\right\} + \operatorname{rot}\{\tilde{\boldsymbol{A}}_t + \boldsymbol{M}\},$$

where $-\boldsymbol{f} = \nabla U + \operatorname{rot}\boldsymbol{M}$; $\tilde{\boldsymbol{A}}_t$ and $\boldsymbol{A}_t$ are the time derivatives of the vectors $\tilde{\boldsymbol{A}}$ and $\boldsymbol{A}$, respectively. Hence, when the velocities and physical density of phases coincide, we obtain $\tilde{\boldsymbol{W}} = \boldsymbol{W}$, $\tilde{\boldsymbol{V}} = \boldsymbol{V}$ and, as consequence, the formulas for the vector fields $\boldsymbol{W}$, $\boldsymbol{V}$ from paper [2]. Thus, the solution $(\boldsymbol{v}, \tilde{\boldsymbol{v}}, p)$ of the system of the two-velocity hydrodynamics equations for homogeneous incompressible media gives a representation of the vector fields $\boldsymbol{W}$, $\boldsymbol{V}$, $\tilde{\boldsymbol{W}}$, $\tilde{\boldsymbol{V}}$, defined in Theorem 1 (where $\boldsymbol{v} = \operatorname{rot}\boldsymbol{A}$, $\tilde{\boldsymbol{v}} = \operatorname{rot}\tilde{\boldsymbol{A}}$) in the form of the sum $\nabla\Phi + \operatorname{rot}\boldsymbol{\Psi}$.

From (13), (15) and Theorem 2 follows

Theorem 4. *For any motion of an ideal two-velocity system with one pressure* ($\boldsymbol{v} \neq 0$, $\tilde{\boldsymbol{v}} \neq 0$)*, the following identities are valid*:

$$\operatorname{div}\boldsymbol{G} = -2\frac{\sin\theta}{v}(\boldsymbol{v}\cdot(\nabla\alpha\times\nabla\theta)), \quad \operatorname{div}\tilde{\boldsymbol{G}} = -2\frac{\sin\tilde{\theta}}{\tilde{v}}(\tilde{\boldsymbol{v}}\cdot(\nabla\tilde{\alpha}\times\nabla\tilde{\theta})).$$

In addition to the general conservation law of Theorem 2, which holds for any smooth vector fields $\boldsymbol{v}(x, y, z, t)$, $\tilde{\boldsymbol{v}}(x, y, z, t)$, *the conservation laws of differential forms are also valid*:

$$\operatorname{div}(\boldsymbol{G} + \boldsymbol{H}_i) = 0, \quad \operatorname{div}(\tilde{\boldsymbol{G}} + \tilde{\boldsymbol{H}}_i) = 0$$

as well as the integrated forms for the fluxes:

$$\int_S([\boldsymbol{G} + \boldsymbol{H}_i]\cdot\boldsymbol{\eta})\,dS = 0, \quad \int_S([\tilde{\boldsymbol{G}} + \tilde{\boldsymbol{H}}_i]\cdot\boldsymbol{\eta})\,dS = 0, \quad i = 1, 2.$$

Here the vectors $\boldsymbol{H}_i$ ($\tilde{\boldsymbol{H}}_i$) *are defined in Theorem 1 and expressed only through the angles* α ($\tilde{\alpha}$), θ ($\tilde{\theta}$) *of the directions of the velocities* $\boldsymbol{v}(x, y, z, t)$ ($\tilde{\boldsymbol{v}}(x, y, z, t)$), S *is a piecewise smooth boundary of the domain* D, $\boldsymbol{\eta}$ *is the normal to* S.

For the irrotational motion (at $\boldsymbol{v} = \nabla u$, $\tilde{\boldsymbol{v}} = \nabla\tilde{u}$), we have

$$\boldsymbol{G} = \frac{1}{v^2}\left\{\nabla u_t + \Delta u\nabla u + \frac{\nabla p}{\bar{\rho}} - \frac{\tilde{\rho}}{2\bar{\rho}}\nabla(\nabla\tilde{u} - \nabla u)^2 - \boldsymbol{f}\right\},$$
$$\tilde{\boldsymbol{G}} = \frac{1}{\tilde{v}^2}\left\{\nabla\tilde{u}_t + \Delta\tilde{u}\nabla\tilde{u} + \frac{\nabla p}{\bar{\rho}} + \frac{\rho}{2\bar{\rho}}\nabla(\nabla\tilde{u} - \nabla u)^2 - \boldsymbol{f}\right\}$$

and the following identities hold:

$$\operatorname{div}\boldsymbol{G} = \frac{2}{v}\operatorname{div}\{u\operatorname{rot}(\alpha\,\nabla\cos\theta)\} = -\frac{2\sin\theta}{v}\frac{\partial(u, \alpha, \theta)}{\partial(x, y, z)},$$

$$\operatorname{div}\tilde{\boldsymbol{G}} = \frac{2}{\tilde{v}}\operatorname{div}\{\tilde{u}\ \operatorname{rot}(\tilde{\alpha}\,\nabla\cos\tilde{\theta})\} = -\frac{2\sin\tilde{\theta}}{\tilde{v}}\frac{\partial(\tilde{u},\tilde{\alpha},\tilde{\theta})}{\partial(x,y,z)}.$$

If one of the following conditions is fulfilled: $u = u(x,y)$ ($\tilde{u} = \tilde{u}(x,y)$) $\Rightarrow$ $\theta \equiv \pi/2$ ($\tilde{\theta} \equiv \pi/2$); $u = u(\alpha,\theta)$ ($\tilde{u} = \tilde{u}(\alpha,\theta)$); $v = v(\alpha,\theta)$ ($\tilde{v} = \tilde{v}(\alpha,\theta)$); $u_z = \varphi(u_x,u_y)$ ($\tilde{u}_z = \tilde{\varphi}(\tilde{u}_x,\tilde{u}_y)$), then $\operatorname{div}\boldsymbol{G} = 0$ ($\operatorname{div}\tilde{\boldsymbol{G}} = 0$).

In the plane case $\boldsymbol{v} = \boldsymbol{v}(x,y,t) = v\,\boldsymbol{\tau}$, $\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}(x,y,t) = \tilde{v}\,\tilde{\boldsymbol{\tau}}$, $\boldsymbol{\tau} = \cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{j}$, $\tilde{\boldsymbol{\tau}} = \cos\tilde{\alpha}\,\boldsymbol{i} + \sin\tilde{\alpha}\,\boldsymbol{j}$, $\alpha = \alpha(x,y,t)$, $\tilde{\alpha} = \tilde{\alpha}(x,y,t)$ is the angle of slope of the line of current (a vector line of the field $\boldsymbol{v}$ ($\tilde{\boldsymbol{v}}$) at $t = \text{const}$). For an incompressible medium we have $\operatorname{div}\boldsymbol{v} = 0$, $\operatorname{div}\tilde{\boldsymbol{v}} = 0$, $\boldsymbol{v} = u_y\boldsymbol{i} - u_x\boldsymbol{j} = \operatorname{rot}(u\,\boldsymbol{k})$, $\tilde{\boldsymbol{v}} = \tilde{u}_y\boldsymbol{i} - \tilde{u}_x\boldsymbol{j} = \operatorname{rot}(\tilde{u}\,\boldsymbol{k})$, $v^2 = u_x^2 + u_y^2$, $\tilde{v}^2 = \tilde{u}_x^2 + \tilde{u}_y^2$, where $u = u(x,y,t)$ and $\tilde{u} = \tilde{u}(x,y,t)$ is a stream function.

From (13), (15) and Theorem 3 follows

Theorem 5. *A system of equations of the two-velocity hydrodynamics with one pressure* (10), (11) *for a plane motion* ($\boldsymbol{v} = \boldsymbol{v}(x,y,t)$, $\tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}(x,y,t)$, $v \neq 0$, $\tilde{v} \neq 0$) *is representable in the form of the identity*

$$\begin{gathered}\boldsymbol{G} = \operatorname{rot}(\alpha\,\boldsymbol{k}), \quad \tilde{\boldsymbol{G}} = \operatorname{rot}(\tilde{\alpha}\,\boldsymbol{k}) \quad \Rightarrow \quad \operatorname{div}\boldsymbol{G} = 0, \quad \operatorname{div}\tilde{\boldsymbol{G}} = 0,\\ \operatorname{rot}\boldsymbol{G} = -(\Delta\alpha)\boldsymbol{k}, \quad \operatorname{rot}\tilde{\boldsymbol{G}} = -(\Delta\tilde{\alpha})\boldsymbol{k} \quad \Rightarrow\\ \Delta\ln v = \operatorname{div}\boldsymbol{Q}, \quad \Delta\ln\tilde{v} = \operatorname{div}\tilde{\boldsymbol{Q}},\\ (\Delta\alpha)\boldsymbol{k} = -\operatorname{rot}\boldsymbol{Q}, \quad (\Delta\tilde{\alpha})\boldsymbol{k} = -\operatorname{rot}\tilde{\boldsymbol{Q}},\end{gathered} \tag{16}$$

where the fields $\boldsymbol{G}$, $\boldsymbol{Q}$, $\tilde{\boldsymbol{G}}$, $\tilde{\boldsymbol{Q}}$ *are defined in* (8), (13), (15).

Remark. From Theorem 3 follows that for the irrotational motion ($\boldsymbol{v} = \nabla u(x,y,t)$, $\tilde{\boldsymbol{v}} = \nabla\tilde{u}(x,y,t)$) with the potentials $u,\tilde{u} \in C^3(D)$ and, in the case of plane, for the motion of an incompressible two-velocity continuum ($\boldsymbol{v} = \operatorname{rot}(u(x,y,t)\boldsymbol{k}) = u_y\boldsymbol{i} - u_x\boldsymbol{j}$, $\tilde{\boldsymbol{v}} = \operatorname{rot}(\widetilde{u(x,y,t)}\boldsymbol{k}) = \tilde{u}_y\boldsymbol{i} - \tilde{u}_x\boldsymbol{j}$) with the stream functions $u,\tilde{u} \in C^3(D)$ for the values α_x, α_y, $v = |\boldsymbol{v}|$, $\boldsymbol{Q}$, $\boldsymbol{S}$, $\boldsymbol{V} = -v^2\boldsymbol{S}$, $\operatorname{div}\boldsymbol{V}$, $\operatorname{rot}\boldsymbol{V}$ ($\tilde{\alpha}_x$, $\tilde{\alpha}_y$, $\tilde{v} = |\tilde{\boldsymbol{v}}|$, $\tilde{\boldsymbol{Q}}$, $\tilde{\boldsymbol{S}}$, $\tilde{\boldsymbol{V}} = -\tilde{v}^2\tilde{\boldsymbol{S}}$, $\operatorname{div}\tilde{\boldsymbol{V}}$, $\operatorname{rot}\tilde{\boldsymbol{V}}$) we obtain the same expressions through the derivative functions "u ($\tilde{u}$)", thus $v = \sqrt{u_x^2 + u_y^2}$, $\tilde{v} = \sqrt{\tilde{u}_x^2 + \tilde{u}_y^2}$, $\boldsymbol{Q} = \frac{\Delta u\nabla u}{v^2}$, $\boldsymbol{S} = \operatorname{rot}(\alpha\,\boldsymbol{k})$, $\tilde{\boldsymbol{Q}} = \frac{\Delta\tilde{u}\nabla\tilde{u}}{\tilde{\boldsymbol{v}}^2}$, $\tilde{\boldsymbol{S}} = \operatorname{rot}(\tilde{\alpha}\,\boldsymbol{k})$,

$$\begin{aligned}\boldsymbol{V} &= \frac{1}{2}\nabla(u_x^2 + u_y^2) - \Delta u\nabla u = -(u_x^2 + u_y^2)\operatorname{rot}(\alpha\boldsymbol{k})\\ &= (u_y u_{xy} - u_x u_{yy})\boldsymbol{i} + (u_x u_{xy} - u_y u_{xx})\boldsymbol{j} = (\nabla u \times \nabla)\nabla u,\end{aligned} \tag{17}$$

$$\operatorname{div}\boldsymbol{V} = 2(u_{xy}^2 - u_{xx}u_{yy}), \quad \operatorname{rot}\boldsymbol{V} = -\{u_y(\Delta u)_x - u_x(\Delta u)_y\}\boldsymbol{k}, \tag{18}$$

$$\begin{aligned}\tilde{\boldsymbol{V}} &= \frac{1}{2}\nabla(\tilde{u}_x^2 + \tilde{u}_y^2) - \Delta\tilde{u}\nabla\tilde{u} = -(\tilde{u}_x^2 + \tilde{u}_y^2)\operatorname{rot}(\tilde{\alpha}\boldsymbol{k})\\ &= (\tilde{u}_y\tilde{u}_{xy} - \tilde{u}_x\tilde{u}_{yy})\boldsymbol{i} + (\tilde{u}_x\tilde{u}_{xy} - \tilde{u}_y\tilde{u}_{xx})\boldsymbol{j} = (\nabla\tilde{u} \times \nabla)\nabla\tilde{u},\end{aligned} \tag{19}$$

$$\operatorname{div}\tilde{\boldsymbol{V}} = 2(\tilde{u}_{xy}^2 - \tilde{u}_{xx}\tilde{u}_{yy}), \quad \operatorname{rot}\tilde{\boldsymbol{V}} = -\{\tilde{u}_y(\Delta\tilde{u})_x - \tilde{u}_x(\Delta\tilde{u})_y\}\boldsymbol{k}, \tag{20}$$

and the following identities hold ($v \neq 0$, $\tilde{v} \neq 0$):

$$\boldsymbol{Q} = \frac{\Delta u \nabla u}{v^2} = \nabla \ln v + \operatorname{rot}(\alpha \boldsymbol{k}), \quad \tilde{\boldsymbol{Q}} = \frac{\Delta\tilde{u}\nabla\tilde{u}}{\tilde{v}^2} = \nabla\ln\tilde{v} + \operatorname{rot}(\tilde{\alpha}\boldsymbol{k}) \;\Leftrightarrow$$
$$\frac{\Delta u}{v^2}\operatorname{rot}(u\boldsymbol{k}) = -\nabla\alpha + \operatorname{rot}(\ln v\boldsymbol{k}), \quad \frac{\Delta\tilde{u}}{\tilde{v}^2}\operatorname{rot}(\tilde{u}\boldsymbol{k}) = -\nabla\tilde{\alpha} + \operatorname{rot}(\ln\tilde{v}\boldsymbol{k}) \;\Rightarrow$$
$$\Delta\ln v = \operatorname{div}\boldsymbol{Q}, \quad \Delta\ln\tilde{v} = \operatorname{div}\tilde{\boldsymbol{Q}},$$
$$(\Delta\alpha)\boldsymbol{k} = -\operatorname{rot}\boldsymbol{Q}, \quad (\Delta\tilde{\alpha})\boldsymbol{k} = -\operatorname{rot}\tilde{\boldsymbol{Q}}.$$

From (12), (14) and (18), (20) follows

Theorem 6. *The system of the Monge–Ampere equations*:

$$u_{xy}^2 - u_{xx}u_{yy} = F, \quad \tilde{u}_{xy}^2 - \tilde{u}_{xx}\tilde{u}_{yy} = \tilde{F}, \tag{21}$$

(*in the general case F and $\tilde{F}$ are smooth functions of the variables x, y, $u, \tilde{u}$, $u_x, \tilde{u}_x$, $u_y, \tilde{u}_y$, $u_{xx}, \tilde{u}_{xx}$, $u_{xy}, \tilde{u}_{xy}$, $u_{yy}, \tilde{u}_{yy}$ and the parameter t*) *and the system of equations for the stream function of a plane motion of incompressible media*

$$\begin{aligned} -\{u_y(\Delta u)_x - u_x(\Delta u)_y\} &= (\Delta u)_t + (\operatorname{rot}\boldsymbol{f}_1^* \cdot \boldsymbol{k}), \\ -\{\tilde{u}_y(\Delta\tilde{u})_x - \tilde{u}_x(\Delta\tilde{u})_y\} &= (\Delta\tilde{u})_t + (\operatorname{rot}\boldsymbol{f}_2^* \cdot \boldsymbol{k}), \end{aligned} \tag{22}$$

are related to each other as follows: *their left-hand sides are expressed, respectively, through divergence and a rotor of the same vector fields $\boldsymbol{V}$, $\tilde{\boldsymbol{V}}$ of the form of* (17), (19) *by formulas* (18), (20), *where $\boldsymbol{f}_1^* = \boldsymbol{f} - \frac{\nabla p}{\bar{\rho}} + \frac{\tilde{\rho}}{2\bar{\rho}}\nabla w$, $\boldsymbol{f}_2^* = \boldsymbol{f} - \frac{\nabla p}{\bar{\rho}} - \frac{\rho}{2\bar{\rho}}\nabla w$, $w = (\tilde{u}_x - u_x)^2 + (\tilde{u}_y - u_y)^2$.*

Let the functions $\boldsymbol{v}(x,y,t) = u_y\boldsymbol{i} - u_x\boldsymbol{j}$, $\tilde{\boldsymbol{v}}(x,y,t) = \tilde{u}_y\boldsymbol{i} - \tilde{u}_x\boldsymbol{j}$, $p(x,y,t)$ in the domain $\Sigma = \{(x,y) \in D, t \in (t_1,t_2)\}$ satisfy a system of equations of the two-velocity hydrodynamics with one pressure (10), (11) *for a plane motion of incompressible media. In this case, in the domain Σ the stream functions $u(x,y,t)$, $\tilde{u}(x,y,t)$ satisfy both equations* (21) *and* (22) *at*

$$F = \frac{\operatorname{div}\boldsymbol{f}_1^*}{2}, \quad \tilde{F} = \frac{\operatorname{div}\boldsymbol{f}_2^*}{2}. \tag{23}$$

Otherwise, let the functions $u(x,y,t)$, $\tilde{u}(x,y,t)$, $p(x,y,t)$ satisfy in the domain Σ equations (21) *and* (22) *with the right-hand side* (23), *and on the boundary S of the domain D at $t \in (t_1,t_2)$, the equality $(\boldsymbol{V}\cdot\boldsymbol{\eta}) = ([\boldsymbol{f}_1^* - \operatorname{rot}(u_t\boldsymbol{k})]\cdot\boldsymbol{\eta})$, $(\tilde{\boldsymbol{V}}\cdot\boldsymbol{\eta}) = ([\boldsymbol{f}_2^* - \operatorname{rot}(\tilde{u}_t\boldsymbol{k})]\cdot\boldsymbol{\eta})$ holds, where $\boldsymbol{\eta}$ is normal to S. In particular, the latter equalities are valid if on the boundary S equalities* (10), (11) *are valid. In this case, the functions $\boldsymbol{v}(x,y,t) = u_y\boldsymbol{i} - u_x\boldsymbol{j}$, $\tilde{\boldsymbol{v}}(x,y,t) = \tilde{u}_y\boldsymbol{i} - \tilde{u}_x\boldsymbol{j}$, $p(x,y,t)$ in the domain Σ satisfy the system of the two-velocity hydrodynamics equations with one pressure* (10), (11) *for the plane motion of incompressible media.*

In particular, for homogeneous media (ρ = const, $\tilde{\rho}$ = const) and a potential field $\boldsymbol{f} = -\nabla U$, equations (21) and (22) take the form

$$\begin{aligned}(\operatorname{rot}\boldsymbol{V}\cdot\boldsymbol{k}) &= -\{u_y(\Delta u)_x - u_x(\Delta u)_y\} = (\Delta u)_t,\\ (\operatorname{rot}\tilde{\boldsymbol{V}}\cdot\boldsymbol{k}) &= -\{\tilde{u}_y(\Delta\tilde{u})_x - \tilde{u}_x(\Delta\tilde{u})_y\} = (\Delta\tilde{u})_t,\end{aligned} \tag{24}$$

$$\frac{\operatorname{div}\boldsymbol{V}}{2} = u_{xy}^2 - u_{xx}u_{yy} = F, \quad \frac{\operatorname{div}\tilde{\boldsymbol{V}}}{2} = \tilde{u}_{xy}^2 - \tilde{u}_{xx}\tilde{u}_{yy} = \tilde{F}, \tag{25}$$

$$F = -\frac{1}{2}\Delta\Big(U + \frac{p}{\bar{\rho}} - \frac{\tilde{\rho}}{2\bar{\rho}}w\Big), \quad \tilde{F} = -\frac{1}{2}\Delta\Big(U + \frac{p}{\bar{\rho}} + \frac{\rho}{2\bar{\rho}}w\Big).$$

Hence, the stream functions $u(x, y, t)$, $\tilde{u}(x, y, t)$, found, for example, as the solution to the known system of equations (24) at any fixed t give simultaneously the solution to a system of the Monge–Ampere equations (25), whose right-hand sides can be found from the system of the two-velocity hydrodynamics equations with one pressure (10), (11) at $\boldsymbol{v} = u_y\boldsymbol{i} - u_x\boldsymbol{j}$, $\tilde{\boldsymbol{v}} = \tilde{u}_y\boldsymbol{i} - \tilde{u}_x\boldsymbol{j}$.

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