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Modeling the 2D seismic waves propagation from singular sources in porous media based on the Laguerre spectral method*

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Abstract. A linear two-dimensional problem in the form of dynamic equations of porous media for the components of velocities, stresses and pressure is considered. Dynamic equations are based on conservation laws and are consistent with the thermodynamics conditions. The medium is considered to be ideal (there is no energy loss in the system) isotropic and two-dimensional inhomogeneous with respect to space. For the numerical solution of the problem posed, the method of integrating the integral Laguerre transform with respect to time with finite-difference approximation in spatial coordinates is used. The solution algorithm employed makes it possible to efficiently carry out simulations in a complex porous medium and to study the wave effects arising in such media.

Introduction

Studying of the processes of convective heat and mass transfer in saturated porous media conventionally occupy one of the central places among modern problems of the theoretical thermal physics. This is primarily due to the relevance of studying the internal mechanisms of mass and energy transfer in a porous medium, including predictions and assessment of the effectiveness of the use of porous materials in various fields of engineering and technology. Porous media are widespread and are diverse both in natural and artificial materials. Therefore, the study of filtration processes takes an important place in biology, hydrology, hydrodynamics, as well as in mechanical engineering, the production of composite materials [1–5] and others.

The first publication to formulate the problem of studying the features of macroscopic mass transfer in a porous medium saturated with a liquid was a report on the experimental research by a graduate of the Ecole Polytechnique, France engineer G. Darcy, published in Paris in the middle of the 19th century [6]. A year later, Darcy published a theoretical study with an analysis of experimental data and the derivation of the known relationship between the velocity of the saturating fluid and the pressure (or head) gradient in a porous medium, which was later named after him [7]. The fundamental nature of the approach and a detailed analysis of the questions

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posed in these publications have a solid basis for a new branch of hydrodynamics — the theory of filtration of liquids and gases in a capillary-porous medium.

When modeling the propagation of seismic waves in a porous medium, the Frenkel-Biot model is often used [8, 9]. Later, a thermodynamically consistent nonlinear mathematical model was proposed for describing elastically deformable processes in a porous medium. It was proposed in [10] based on first general physical principles. The peculiarity of the models discussed is the existence of three types of sound vibrations: transverse and two types of longitudinal. In contrast to models of the Frenkel-Biot type, in the linearized model [10], the medium is described by three elastic parameters [11, 12]. These elastic parameters are one-to-one expressed by the three velocities of elastic vibrations. This circumstance is important for the numerical simulation of the propagation of elastic waves in porous media, when the distributions of the velocities of acoustic waves and physical densities of the matrix, saturating a fluid, and porosity are known.

In this paper, we numerically solve a system of linearized equations for porous media from [11, 12] in the absence of energy dissipation in the two-dimensional case. The original system is written down as a hyperbolic system in terms of matrix velocities, saturating fluid velocity, stress tensor and fluid pressure. For the numerical solution of the problem posed, the method of combining the analytical transformation and the finite-difference method is used. The algorithm proposed is based on the use of the integral Laguerre transform with respect to the time coordinate. This method can be considered as an analogue of the well-known spectral method based on the Fourier transform. However, in contrast to it, the use of the integral Laguerre transform in time allows us to reduce the original problem to solving a system of equations in which the separation parameter is present only in the right-hand side of the equations and has a recurrent dependence. This method for solving dynamic problems of the elasticity theory was first considered in [13, 14] and then developed for problems of viscoelasticity [15, 16]. In these published works, the distinctive features of this method from the accepted approaches are considered, and the advantages of using the integral Laguerre transform in contrast to the difference method and the Fourier transform with respect to time are discussed. In particular there, it is shown that this solution algorithm is effective in modeling the wave processes in media with sharply contrasting boundaries, such as earth-water-atmosphere.

1. Formulation of the problem

Let us consider the formulation of the dynamic problem of the propagation of seismic waves from singular sources in media consisting of elastic and porous layers. In this case, the propagation of seismic waves in a porous medium saturated with a liquid in the absence of energy loss is described for the Cartesian coordinate system in a half-plane $x_2 \geq 0$ by the following initial-boundary value problem [11, 12, 17]:

$$
\begin{gathered}
\frac{\partial u_i}{\partial t} + \frac{1}{\rho_{0,s}}\frac{\partial \sigma_{ik}}{\partial x_k} + \frac{1}{\rho_0}\frac{\partial p}{\partial x_i} = F_i(x_1, x_2) f(t), \\
\frac{\partial v_i}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x_i} = F_i(x_1, x_2) f(t), \\
\frac{\partial \sigma_{ik}}{\partial t} + \mu\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right) + \left(\frac{\rho_{0,l}}{\rho_0}K - \frac{2}{3}\mu\right)\delta_{ik} \operatorname{div} \vec{u} - \frac{\rho_{0,s}}{\rho_0} K \delta_{ik} \operatorname{div} \vec{v} = 0, \\
\frac{\partial p}{\partial t} - (K - \alpha\rho_0\rho_{0,s}t) \operatorname{div} \vec{u} + \alpha\rho_0\rho_{0,l} \operatorname{div} \vec{v} = 0, \\
u_i|_{t=0} = v_i|_{t=0} = \sigma_{ik}|_{t=0} = p|_{t=0} = 0, \\
\sigma_{22} + p|_{x_2=0} = \sigma_{12}|_{x_2=0} = \frac{\rho_{0,l}}{\rho_0} p \,\Big|_{x_2=0} = 0,
\end{gathered} \tag{1}
$$

where $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ are the velocity vectors of an elastic porous body with a partial density $\rho_{0,s}$ and a liquid with a partial density $\rho_{0,l}$, respectively, p is the pore pressure, σ_{ik} is the stress tensor $\rho_0 = \rho_{0,l} + \rho_{0,s}$, $\rho_{0,s} = \rho^f_{0,s}(1 - d_0)$, $\rho_{0,l} = \rho^f_{0,l} d_0$, $\rho^f_{0,s}$, and $\rho^f_{0,l}$ are the physical densities of the elastic porous body and the liquid, respectively, d_0 is the porosity, δ_{ik} is the Kronecker symbol, $K = \lambda + 2\mu/3$, $\lambda > 0$, $\mu > 0$ are the Lame coefficients, $\alpha = \rho_0\alpha_3 + K/\rho_0^2$, $\rho_0^3\alpha_3 > 0$ is the modulus of volumetric compression of the liquid component of the heterophase medium, $\vec{F} = (F_1, F_2)$ is the vector of mass forces, $f(t)$ is the simulated time signal in the source. F_1 and F_2 are the components of the force vector describing the action of a source localized in space. The values of these components depend on the type of the simulated source:

- For a source of the “vertical force” type
  $F_1 = 0,\ F_2 = \delta(x_1 - x_0)\,\delta(x_2 - z_0)$;
- For a source of the “center of pressure” type
  $F_1 = \delta(x_2 - z_0)\dfrac{\partial\delta(x_1 - x_0)}{\partial x_1},\ F_2 = \delta(x_1 - x_0)\dfrac{\partial\delta(x_2 - z_0)}{\partial x_2}$;
- for a source of the “dipole without moment” type
  $F_1 = 0,\ F_2 = \delta(x_1 - x_0)\dfrac{\partial\delta(x_2 - z_0)}{\partial x_2}$.

Here x_0, z_0 are the spatial coordinates of the source.

Elastic moduli K, μ, α_3 are expressed through the velocity of propagation of the shear wave c_s and the two velocities of the longitudinal waves c_{p_1}, c_{p_2} — by the following formulas [18, 19]:

$$\mu = \rho_{0,s} c_s^2,$$

$$K = \frac{\rho_0}{2} \frac{\rho_{0,s}}{\rho_{0,l}} \left(c_{p_1}^2 + c_{p_2}^2 - \frac{8}{3} \frac{\rho_{0,l}}{\rho_0} c_s^2 - \sqrt{(c_{p_1}^2 - c_{p_2}^2)^2 - \frac{64}{9} \frac{\rho_{0,l}\rho_{0,s}}{\rho_0^2} c_s^4} \right),$$

$$\alpha_3 = \frac{1}{2\rho_0^2} \left(c_{p_1}^2 + c_{p_2}^2 - \frac{8}{3} \frac{\rho_{0,s}}{\rho_0} c_s^2 + \sqrt{(c_{p_1}^2 - c_{p_2}^2)^2 - \frac{64}{9} \frac{\rho_{0,l}\rho_{0,s}}{\rho_0^2} c_s^4} \right).$$

2. The solution algorithm

To solve problem (1), we apply the integral Laguerre transform with respect to time [14, 15]:

$$\overrightarrow{W}_m(x_1, x_2) = \int_0^\infty \overrightarrow{W}(x_1, x_2, t)(ht)^{-\alpha/2} l_m^\alpha(ht) d(ht),$$

with the inversion formulas

$$\overrightarrow{W}(x_1, x_2, t) = (ht)^{\alpha/2} \sum_{m=0}^{\infty} \frac{m!}{(m+\alpha)!} \overrightarrow{W}_m(x_1, x_2) l_m^\alpha(ht),$$

where $l_m^\alpha(ht)$ are the Laguerre functions.

As a result of this transformation, the original problem (1) is reduced to a two-dimensional spatial differential problem in the spectral domain:

$$\frac{h}{2} u_i^m + \frac{1}{\rho_s} \frac{\partial \sigma_{ik}^m}{\partial x_k} + \frac{1}{\rho_0} \frac{\partial P^m}{\partial x_i} = F_i^m(x_1, x_2) - h \sum_{n=0}^{m-1} u_i^n,$$

$$\frac{h}{2} v_i^m + \frac{1}{\rho_0} \frac{\partial P^m}{\partial x_i} = F_i^m(x_1, x_2) - h \sum_{n=0}^{m-1} v_i^n,$$

$$\frac{h}{2} \sigma_{ik}^m + \mu \left(\frac{\partial u_k^m}{\partial x_i} + \frac{\partial u_i^m}{\partial x_k} \right) + \left(\lambda - \frac{\rho_s}{\rho_0} K \right) \delta_{ik} \operatorname{div} \vec{u}^m - \frac{\rho_s}{\rho_0} K \delta_{ik} \operatorname{div} \vec{v}^m = -h \sum_{n=0}^{m-1} \sigma_{ik}^n,$$

$$\frac{h}{2} P^m - (K - \alpha\rho_0\rho_s) \operatorname{div} \vec{u}^m + \alpha\rho_0\rho_l \operatorname{div} \vec{v}^m = -h \sum_{n=0}^{m-1} P^n,$$

$$\sigma_{22}^m + P^m|_{x_2=0} = \sigma_{12}^m|_{x_2=0} = \frac{\rho_l}{\rho_0} P^m \bigg|_{x_2=0} = 0. \qquad (2)$$

To solve the above problem, we will use the finite difference approximation of derivatives with respect to spatial coordinates on staggered grids with the 4th order of accuracy [20]. To do this, in the computational domain in the direction of the coordinate $z = x_1$, the grids ωz_1 and $\omega z_{1/2}$ with a discretization step Δz are shifted relative to each other by $\Delta z/2$:

$$\omega z_1 = (x, j\Delta z, t), \quad \omega z_{1/2} = (x, j\Delta z + \Delta z/2, t), \quad j = 0, \ldots, M.$$

Similarly, we introduce in the direction of the coordinate $x = x_2$ the grids ωx_1 and $\omega x_{1/2}$ with the discretization step Δx shifted relative to each other by $\Delta x/2$:

$$\omega x_1 = (i\Delta x, z, t), \quad \omega x_{1/2} = (i\Delta x + \Delta x/2, z, t), \quad i = 0, \ldots, N.$$

On these grids, we introduce the differentiation operators D_x and D_z approximating the derivatives $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial z}$ with the fourth order of accuracy in the coordinates $z = x_1$ and $x = x_2$. Let us define the components of the solution vector at the following grid nodes:

$$\begin{aligned} u_1(x,z),\ \nu_1(x,z) \quad &\text{at} \quad \omega x_1 \times \omega z_1, \\ u_2(x,z),\ \nu_2(x,z) \quad &\text{at} \quad \omega x_{1/2} \times \omega z_{1/2}, \\ \sigma_{11}(x,z),\ \sigma_{22}(x,z),\ P(x,z) \quad &\text{at} \quad \omega x_{1/2} \times \omega z_1, \\ \sigma_{12}(x,z) \quad &\text{at} \quad \omega x_1 \times \omega z_{1/2}. \end{aligned}$$

As a result of the finite-difference approximation of problem (2), we obtain a system of linear algebraic equations. Let us represent the required solution vector W in the following form:

$$W = (V_{0,0}, V_{0,1}, \ldots, V_{M,N})^T,$$

$$V_{i,j} = (u_1^{i,j}, u_2^{i+1/2,j+1/2}, \nu_1^{i,j}, \nu_2^{i+1/2,j+1/2}, \sigma_{11}^{i+1/2,j}, \sigma_{22}^{i+1/2,j}, \sigma_{12}^{i,j+1/2}, P^{i+1/2,j}).$$

Then, this system of linear algebraic equations in the vector form can be written down as

$$\Big(A + \frac{h}{2}E\Big)W^m = F^{m-1}.$$

As a result, the matrix of the system of the reduced problem is well conditioned, which makes it possible to use fast methods for solving systems of linear algebraic equations based on iterative methods, such as the conjugate gradients, converging to the desired solution with the required accuracy in just a few iterations.

3. Numerical results

We present the numerical results of modeling the seismic wave fields for a test medium model consisting of two isotropic layers — the upper layer is water and the lower one is a porous medium. The physical characteristics of the layers were specified by the following parameters:

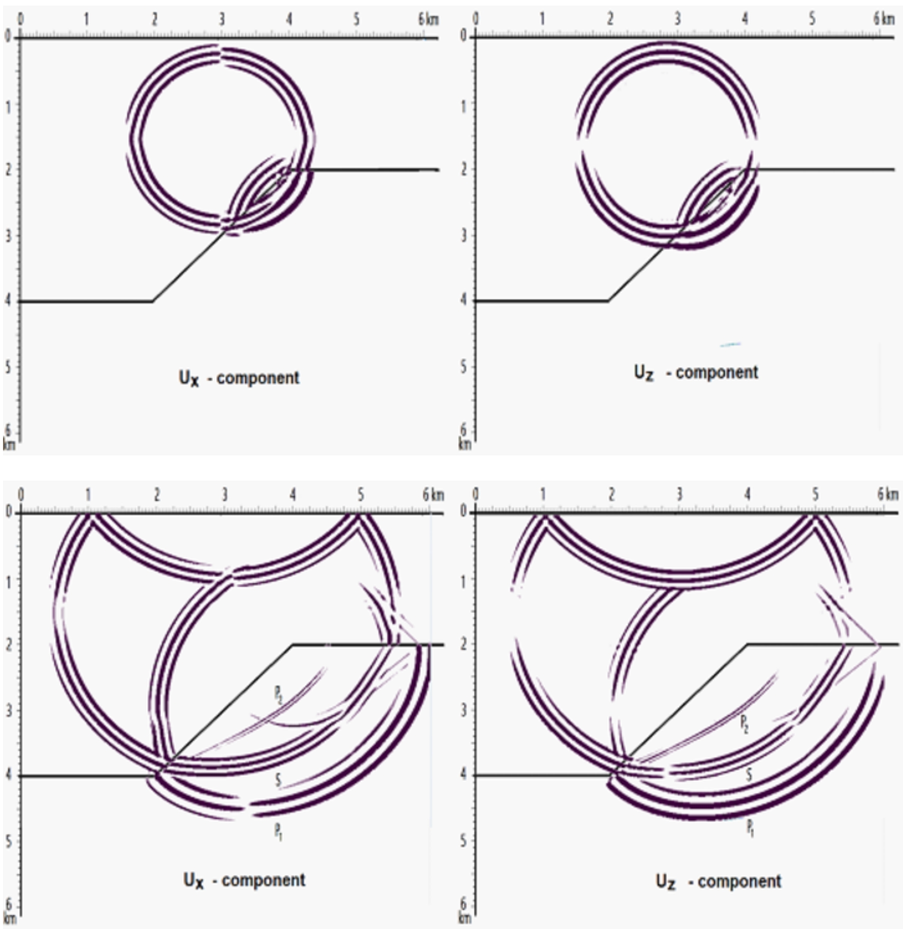
- the upper layer — $\rho = 1$ g/cm^3, $c_p = 1.5$ km/s, $c_s = 0$;

- the bottom layer— $\rho^f_{0,s} = 1.5$ g/sm^3, $\rho^f_{0,l} = 1$ g/sm^3, $c_{p_1} = 2.1$ km/s, $c_{p2} = 0.6$ km/s, $c_s =$ 1.3 km/s, $d = 0.2$.

The wave field was simulated from a point source of the dipole type without a moment with the coordinates $x_0 = 3$ km, $z_0 = 1.5$ km, which is located in the upper water layer. The time signal in the source was set in the form of the Puzyrev pulse with a carrier in the form

$$f(t) = \exp\left(-\frac{2\pi f_0(t-t_0)^2}{\gamma^2}\right)\sin(2\pi f_0(t-t_0)),$$

where $\gamma = 4,\ f_0 = 10$ Hz, $t_0 = 0.15$ s.



Snapshots of the wave field of the displacement velocity at the time instants $T = 1$ (top) and 1.8 (bottom) seconds: the left for the component $u_x(x, z)$, the right for the component $u_z(x, z)$

The results of numerical calculations of the wave field for a given model of the medium are presented in the figure. The interface between the layers is shown by a solid line.

The figure shows that when a longitudinal wave emitted by a source of a given type falls on the interface between the layers, the corresponding types of waves in a given medium are formed. In the water layer, the longitudinal waves reflected from the boundaries appear, and in the lower porous layer, two types of the longitudinal waves P_1 and P_2 and the transverse wave S appear.

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