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Numerical solution of a one-dimensional inverse retrospective problem for a system of poroelasticity equations

P.V. Korobov, V.V. Alekseev

Abstract. This paper deals with the implementation of the statement of the second initial condition for a system of dynamic equations of a two-phase one-dimensional medium. The need to solve such a problem arises, for example, in the case when the time conditions are specified at the initial and final moments of time. The paper describes a numerical model for the direct and inverse problems, gives examples of the numerical results of solving the problems posed.

1. Statement of the problem and description of the solution method

Let us consider the initial-boundary value problem for the system of poroelasticity equations with homogeneous boundary conditions [1–4]:

$$\rho_s u_{tt} = (\mu u_x)_x - \rho_l^2 \chi (u - v)_t, \quad x \in (0, L), \quad t \in (0, T), \tag{1}$$

$$\rho_l v_t = \rho_l^2 \chi (u - v), \quad x \in (0, L), \quad t \in (0, T), \tag{2}$$

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = \phi(x), \quad x \in (0, L), \tag{3}$$

$$v|_{t=0} = 0, \quad x \in (0, L). \tag{4}$$

$$u|_{x=0} = u|_{x=L} = 0, \quad t \in (0, T), \tag{5}$$

Let the function $\phi(x)$ be unknown. Additional information is given at the time instance T:

$$u(x, T) = \tilde{u}(x). \tag{6}$$

In the considered problem for the system of poroelasticity equations, the functions satisfy equations (1), (2), the first initial condition from (3), the initial condition (4), and the zero Dirichlet conditions (5). In the retrospective inverse problem, the functions $u_0(x)$, $\tilde{u}(x)$, $\mu(x) > 0$, $\chi(x) > 0$ and the constants ρ_s, ρ_l are given. The functions $u(x,t)$, $v(x,t)$ and the initial condition $\phi(x)$ are to be determined.

For the numerical solution of the initial-boundary value problem (1)–(5) we use a three-layer difference scheme with the weight factors of the second order of accuracy in t with a step τ and in x with an approximation step h for equation (1). To approximate equation (2), we use a scheme of the first order of accuracy in t [5].

We introduce the grid operator $A = \{A_j\}$:

$$A_j u^i = (\mu_j u^i_{\bar{x},j})_x, \quad \mu_j = \mu((j-1/2)h). \tag{7}$$

Therefore,

$$\frac{1}{\tau^2}(u_j^{i+1} - 2u_j^i + u_j^{i-1}) = \frac{1}{2h^2\rho_s}(\sigma_1 A_j u^{i+1} + (1-2\sigma_1)A_j u^i + \sigma_1 A_j u^{i-1}) - \frac{1}{\tau\rho_s}\rho_l^2 \chi_j((u_j^i - v_j^i) - (u_j^{i-1} - v_j^{i-1})), \tag{8}$$

$$v_j^{i+1} - v_j^i = \rho_l \tau \chi_j \Big(u_j^i - \frac{1}{2}(\sigma_2 v_j^{i+1} + (1-\sigma_2)v_j^i)\Big), \tag{9}$$

for $i = 1, \ldots, N-1$, $j = 1, \ldots, M-1$. Here $\chi_j = \chi(jh)$.

We approximate the initial and boundary conditions with the first order of accuracy:

$$u_j^0 = u_0(jh), \quad \frac{u_j^1 - u_j^0}{\tau} = \phi(jh), \quad j = 0, \ldots, M,$$
$$v_j^0 = 0, \quad j = 0, \ldots, M,$$
$$u_0^i = u_M^i = 0, \quad i = 0, \ldots, N.$$

The stability of the difference approximation of the problem is achieved at $\sigma_1 \geq 1/4$, $\sigma_2 \geq 1/2$ [6]. We use $\sigma_1 = 1/4, \sigma_2 = 1/2$.

Solving the inverse problem is equivalent to solving the operator equation

$$\bar{A}q = \tilde{u}(x),$$

where $\bar{A}$ is the self-adjoint operator of the direct problem.

The problem of determining the function ϕ by the additional condition $u(x,T) = \tilde{u}$ can be reduced to the problem of minimizing the functional $H(q) = (\bar{A}q, q) - 2(\tilde{u}, q)$ [7] if the operator $\bar{A}$ is positive-definite.

The proof of positive definiteness of the operator $\bar{A}$ requires more investigations. We just supposed this and will try the conjugate gradient method to minimize the function $H(q)$.

The iterative conjugate gradient method for determining the functions $u(x,t)$, $v(x,t)$, $\phi(x)$ from equations (1)–(5) and additional condition (6) is as follows:

1. Choose an arbitrary initial approximation $\phi^0(x)$ for the function $\phi(x)$.
2. Next, using the difference approximation (8), (9), we determine the solution of the direct problem $u^0(x,t)$, $v^0(x,t)$ for the approximation $\phi^0(x)$.
3. Choose as the initial residual $r_0(x) = u^0(x,T) - \tilde{u}(x)$ and the auxiliary function $p_0(x) = r_0(x)$.

4. At the kth iteration ($k = 1, 2, \ldots$), we find the solution $\bar{u}^k(x,t)$, $\bar{v}^k(x,t)$ of the direct problem (1)–(5) with $\bar{u}(x,0) = 0$, $\bar{u}_t(x,0) = p_k(x)$ in (3). Then, we obtain $\bar{A}p_k = \bar{u}^k(x,T)$.
5. From the solution obtained, we determine the iterative parameters for the conjugate gradient method by the formulas:

$$\alpha_k = \frac{(r_k, r_k)}{(\bar{A}p_k, p_k)}, \qquad r_{k+1} = r_k - \alpha_k \bar{A}p_k,$$
$$\beta_k = \frac{(r_{k+1}, r_{k+1})}{(r_k, r_k)}, \qquad p_{k+1} = r_k + \beta_k p_k.$$

6. The function ϕ^{k+1} is found as follows:

$$\phi^{k+1} = \phi^k + \alpha_k p_k.$$

The solution $u^{k+1}(x,t)$, $v^{k+1}(x,t)$ is found from the direct problem (1)–(5) with the second initial condition $u_t^{k+1}(x,0) = \phi^{k+1}(x)$.

2. Numerical examples

As examples of the numerical implementation, we consider a straight-line problem (1)–(5) with homogeneous boundary conditions with $\mu(x,t) = 1$, $u_0(x) = 0$, $\chi = \text{const}$, $\rho_l = \rho_s = 1$, $T = L = 1$, $\phi(x) = \pi \sin(\pi x)$.

For solving the inverse problem, we will use (6) with known $\tilde{u}$.

In Figures 1, 2, the solution $u(x,t)$ of the direct problem is shown.

In Figures 3–7, we show the results of restoring the second initial condition ϕ for various initial approximations $\phi^0(x)$ and the solution $u(x,t)$ of the inverse problem with the conjugate gradient method and various parameters of the computational grid.

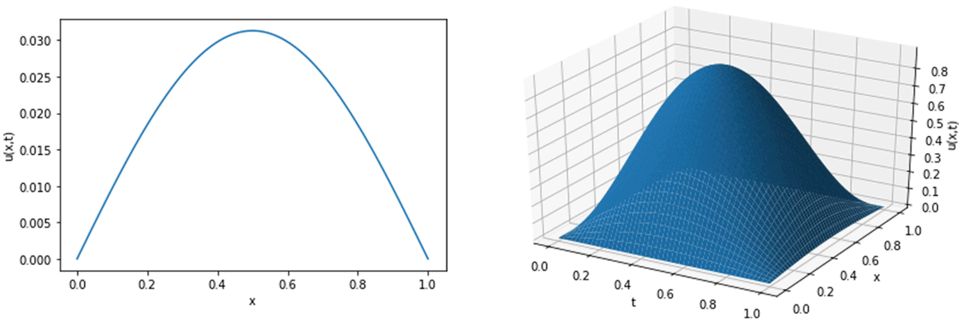

Figure 1. The solution to the direct problem for $\chi = 0.5$: at $T = 1$ (left) and the whole graph (right)

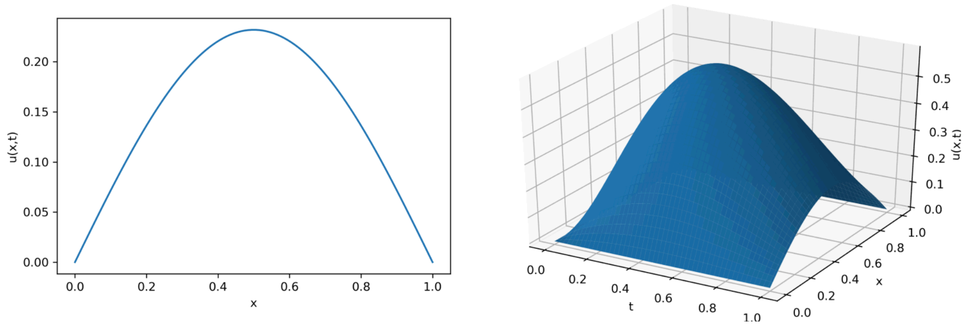

Figure 2. The solution to the direct problem for $\chi = 5$: at $T = 1$ (left) and the whole graph (right)

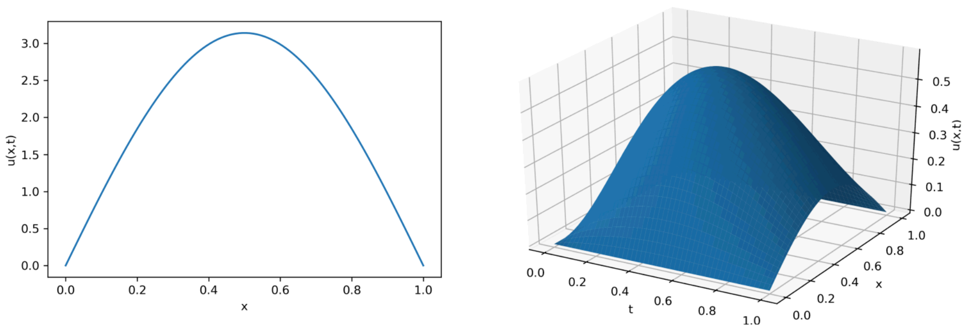

Figure 3. The solution to the inverse problem for $\phi^0(x) = 10\pi \sin(\pi x)$, $\chi = 5$, $N = 100$, $M = 100$: reconstructed function ϕ (left) and the whole graph (right)

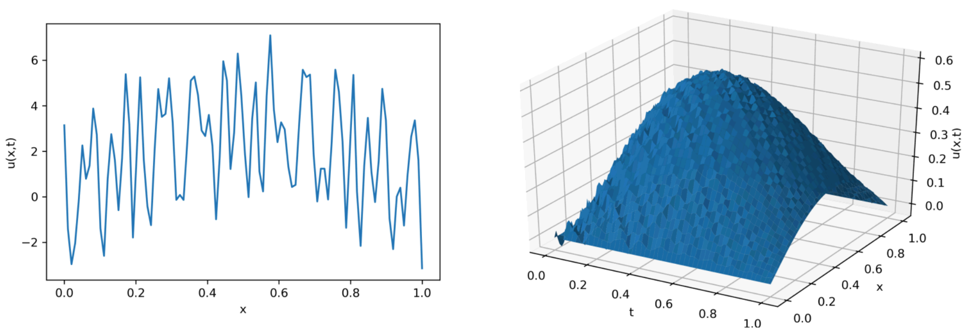

Figure 4. The solution to the inverse problem for $\phi^0(x) = \pi \cos(\pi x)$, $\chi = 5$, $N = 100$, $M = 100$: reconstructed function ϕ (left) and the whole graph (right)

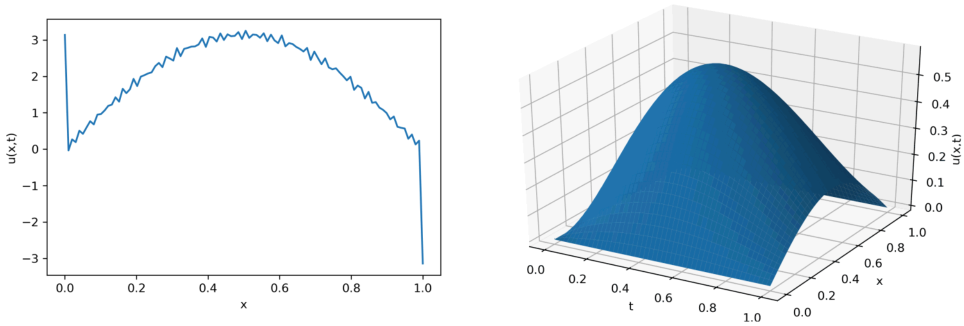

Figure 5. The solution to the inverse problem for $\phi^0(x) = \pi\cos(\pi x)$, $\chi = 5$, $N = 500$, $M = 100$: reconstructed function ϕ (left) and the whole graph (right)

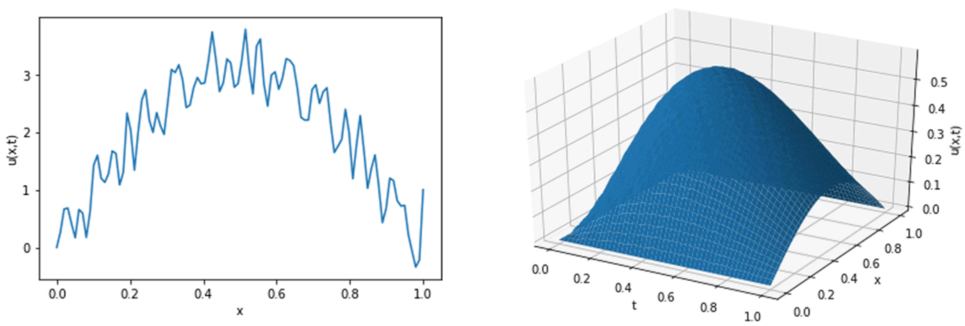

Figure 6. The solution to the inverse problem for $\phi^0(x) = x$, $\chi = 5$, $N = 100$, $M = 100$: reconstructed function ϕ (left) and the whole graph (right)

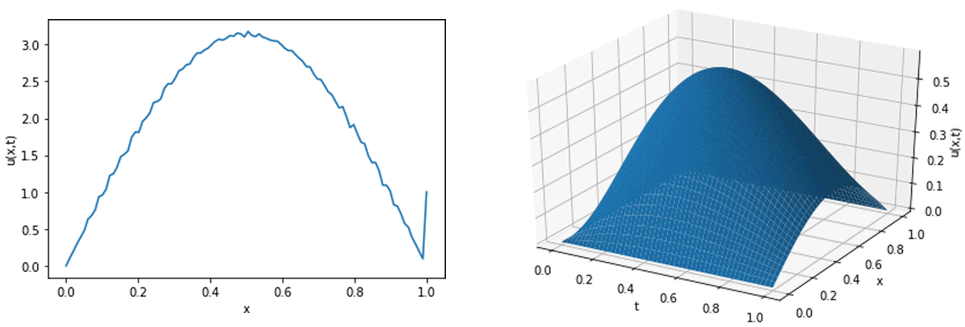

Figure 7. The solution to the inverse problem for $\phi^0(x) = x$, $\chi = 5$, $N = 500$, $M = 100$: reconstructed function ϕ (left) and the whole graph (right)

3. Conclusion

As a result of the study described in this paper, a numerical model was implemented to solve the reverse retrospective problem. The conjugate gradient method was used to minimize the objective functional $H(q)$. The efficiency of the method proposed is shown for various initial approximations to the solution of the problem. The rate of convergence and, accordingly, the computational complexity of the process depends on the proximity of the initial approximation to the exact solution of the problem.

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