Joint NCC & IIS Bull., Comp. Science, 13 (2000), 40–55
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Background for formalisation of complex systems*

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Based on notions of computability for operators and real-valued functionals, a background for formalisation of complex systems is introduced. We propose a recursion scheme which is a suitable tool for formalisation of complex systems, such as hybrid systems. In this framework the trajectories of continuous parts of hybrid systems can be represented by computable functionals.

1. Introduction

Recently more attention is paid to the problems of exact mathematical formalisation of complex systems such as hybrid systems. By a hybrid system we mean a network of digital and analog devices interacting at discrete times. An important characteristic of a hybrid system is that it incorporates continuous components, usually called plants, as well as digital components, i.e. digital computers, sensors and atuators controlled by programs. These programs are designed to select, control, and supervise behaviour of the continuous components. Modelling, design, and investigation of behaviours of hybrid systems have recently become areas of active research in computer science.

The main subject of our investigation is behaviour of the continuous components. In [23], the set of all possible trajectories of the plant was called a performance specification.

We propose a background for formalisation of hybrid systems based on Domain Theory.

Our approach differs from the previous ones in the following: we can characterise the continuous and discret parts, as well as interactions between them in the same algebraic model, with the help of finite formulas. We propose a general approach to formalisation of a hybrid system based on the theory of computability over the reals.

At present, new applications of Domain Theory to computations over various spaces are being developed. Domain Theory was independently introduced by Dana Scott [26] as a mathematical theory of computation in the semantics of programming languages and by Yu. L. Ershov [7] as a theory of partial computable functionals of finite type.

A domain is a partially ordered set equipped with the notions of limit and finite approximation; the partial order corresponds to information on the elements.

Given a computation based on an algorithm, each of the sets of input and output forms a domain. The program which carries out the computation is represented as a function between these domains. Every new step in the computation results in an element of the domain of output which provides more information and better approximation to the ultimate result.

A continuous function is one which preserves the information order (so that more input information gives more output information) and the limits of infinite computations in the domain (so that the total information obtainable as output from an infinite sequence of input elements with refining information is the sum of all information obtained from each input element).

There are a number of categories of domains according to various additional properties that they satisfy (algebraic domains [1, 29, 30], continuous domains [26, 27, 5, 6, 10, 33, 34, 35, 24], and so on). Below, to construct computational models for real-valued functions and Functionals, we will use continuous domains. The continuous domain (more precisely, the interval domain) for the reals was first proposed by Dana Scott [26] and later was applied to mathematics, physics and real-number computation in [5, 6, 34, 35, 24] and others.

*This reseach was supported in part by the RFBR (grants N 99-01-00485, N 00-01-00810) and by the Siberian Division of RAS (a grant for young reseachers, 2000)

In this article we propose continuous domains named function domains to construct a computational model of operators and a real-valued functional defined on the set of continuous real-valued functions.

In Section 2, we recall basic definitions and tools from [6] and introduce new ones to construct our computational model. We introduce effective function domains which are ω-continuous Scott domains. Based on the notion of computability of mapping between two domains, we propose computability of operators and functionals defined on continuous real-valued functions. The main feature of this approach is related to the fact that computable operators and functionals defined on continuous real-valued functions are continuous on their domain w.r.t. the standard topology induced by the uniform norm. Moreover, we propose a semantic characterisation of computable operators and functionals via validity of finite Σ-formulas.

Then, in Section 3, we give characterisations of computable functions and functionals in logical terms via the definability theory. Also we propose a recursion scheme which is a suitable tool for formalisation of complex systems such as hybrid systems. Modelling, design, and investigation of the behaviour of hybrid systems have recently become active areas of research in computer science (for example, see [12, 13, 17, 20, 23]).

In the framework proposed in this paper the trajectories of continuous parts of hybrid systems (performance specifications) can be represented by computable functionals.
For more details we would like to refer to the full version of this paper on http://inet.ssc.nsu.ru/ rita/complex.ps.

2. Basic notions

To propose the notions of computability of operators and real-valued functionals, we, following the paper [6], recall the definitions of the continuous domain for the reals (the interval domain) and computable functions and introduce functional domains.

2.1. Terminology

Throughout the article, $< \mathbf{R}, 0, 1, +, \cdot, <>$ is the standard model of the reals, denoted also by $\mathbf{R}$, where $+$ and $\cdot$ are regarded as predicate symbols. We use the language of strictly ordered rings, so the predicate $<$ positively occurs in formulas.

Let $\mathbf{R}^-$ denote $\mathbf{R} \cup \{-\infty\}$, $\mathbf{R}^+$ denote $\mathbf{R} \cup \{+\infty\}$, $\mathbf{N}$ denote the set of natural numbers and $\mathbf{Q}$ the set of rational numbers.

2.2. The effective interval domain for the reals

The interval domain for the reals was first proposed by Dana Scott [26, 27] and later was applied to mathematics, physics and real number computation (see, for example, [5, 6, 24, 33, 34, 35]). By *the interval domain for the reals* we mean the set of compact intervals of $\mathbf{R}$, partially ordered with the reversed subset inclusion. The real line is obtained as the set of maximal elements in this continuous domain.
We recall the definition of the *interval domain* $\mathcal{I}$ proposed in [6]:

$\mathcal{I} = \{[a, b] \subseteq \mathbf{R} \mid a, b \in \mathbf{R},\ a \leq b\} \cup \{\perp\}$.

The order is the reversed subset inclusion, i.e. $\perp \sqsubseteq I$ for all $I \in \mathcal{I}$ and $[a, b] \sqsubseteq [c, d]$ iff $a \leq c$ and $d \leq b$ in the usual ordering of the reals. One can consider the least element $\perp$ as the set $\mathbf{R}$. Directed suprema are filtered intersections of intervals. The way-below relation is given by $I \ll J$ iff $J \subseteq \mathrm{int}(I)$, where $\mathrm{int}(I)$ denotes the interior of I. For the relation $\ll$ we have the following properties: $\perp \ll J$ for all $J \in \mathcal{I}$ and $[a, b] \ll [c, d]$ if and only if $a < c$ and $b > d$. The maximal elements are the intervals $[a, a]$ denoted as $\{a\}$. Note that $\mathcal{I}$ is an effectively given ω-continuous domain. An example for a countable basis is the collection $\mathcal{I}_0$ of all intervals with rational endpoints together with the least element $\perp$. Similarly, we can define the interval domain $\mathcal{I}_{[a,b]}$ for an interval $[a,b]$.

Definition 2.1. *Let $\mathcal{I}_0 = \{b_0, \dots, b_n, \dots\}$ be the effective enumerated set of all intervals with rational endpoints endowed with the least element $\perp$.*
A continuous function $f : \mathcal{I} \to \mathcal{I}$ is computable, if the relation $b_m \ll f(b_n)$ is computable enumerable in n, m, where b_m, $b_n \in \mathcal{I}_0$.

Definition 2.2. *A function $f : \mathbf{R} \to \mathbf{R}$ is computable if and only if there is an enlargement $g : \mathcal{I} \to \mathcal{I}$ (i.e., $g(\{x\}) = \{f(x)\}$ for all $x \in \mathrm{dom} f$) which is computable in the sense of Definition 2.1.*

Denote the class of computable total functions as $\mathcal{F}$ and the class of computable functions defined on an interval $[a,b]$ as $\mathcal{F}[a,b]$.

2.3. The effective function domain

In this section we introduce effective function domains which are ω-continuous Scott domains. Based on the notion of computability of mapping between two domains, we propose computability of operators and functionals defined on continuous real-valued functions. Sufficiency of this approach follows from the fact that computable operators and functionals defined on continuous real-valued functions are continuous on their domain w.r.t. the standard topology induced by the uniform norm. Moreover, we propose a semantic characterisation of computable operators and functionals via validity of finite Σ-formulas.

We consider the set of functions $f : [a,b] \to \mathcal{I}$ defined on a compact interval $[a,b]$ which are continuous in the following sense.

Definition 2.3. *A function $f : [a,b] \to \mathcal{I}$ is said to be continuous in x_0 if $f(x_0) = \perp$ or $f(x_0) = [c,d]$ and $\forall \epsilon_1 \epsilon_2 \exists \delta \, (|x - x_0| < \delta \to f(x) \gg [c - \epsilon_1, d + \epsilon_2])$.*
A function is continuous on $[a,b]$ if it is continuous in every point of $[a,b]$.

Note that a continuous function $f : [a,b] \to \mathcal{I}$ can be represented by the pair of a lower semicontinuous map and an upper semicontinuous map.

Definition 2.4. *A function $f : [a,b] \to \mathbf{R}^-$ is said to be lower semicontinuous if the set $Y_f = \{x | f(x) \neq -\infty\}$ is open w.r.t. the standard topology and*

$$(\forall x_0 \in Y_f)\, (\forall a < f(x_0))\, \exists \delta \, (|x_0 - x| < \delta \to a < f(x))\,.$$

A function $f : [a,b] \to \mathbf{R}^+$ is said to be upper semicontinuous if the set $Y_f = \{x | f(x) \neq +\infty\}$ is open w.r.t. the standard topology and

$$(\forall x_0 \in Y_f)\, (\forall a > f(x_0))\, \exists \delta \, (|x_0 - x| < \delta \to a > f(x))\,.$$

For the classical theory of semicontinuous functions the reader should consult some textbook (e.g. [3]). The reader can also find some properties of computability on continuous and semicontinuous real functions in [36]. It is easy to see that a continuous function $f : [a,b] \to \mathcal{I}$ is closely related to the pair of functions $\langle f^1 : [a,b] \to \mathbf{R}^-, f^2 : [a,b] \to \mathbf{R}^+ \rangle$, where $f^1(x) = \inf\ f(x)$ is lower semicontinuous and $f^2(x) = \sup\ f(x)$ is upper semicontinuous (see [6, 24]). The function f^1 is called as *lower bound* of f and f^2 is called as *upper bound* of f.

Below we denote $Y_f = \{x | f(x) \neq \perp\}$ for $f : [a,b] \to \mathcal{I}$ and $Y_f = \{x | f(x) \neq \pm\infty\}$ for $f : [a,b] \to \bar{\mathbf{R}}$. For upper and lower semicontinuous functions these sets are open w.r.t. the standard topology by definition. To introduce our notions of computable operators and real-valued functionals, we introduce functional domains which are ω-continuous Scott domains.

Definition 2.5. *A function domain $\mathcal{I}_f([a,b])$ is the collection of all continuous functions $f : [a,b] \to \mathcal{I}$ with a least element $\perp_{[a,b]}$ partially ordered by the following relation: $f \sqsubseteq g$ iff $(\forall x \in [a,b])\,(f(x) \sqsubseteq g(x))$ and $\perp_{[a,b]} \sqsubseteq I$ for all $I \in \mathcal{I}_f([a,b])$.*

To denote that a subset $A \subseteq \mathcal{I}_f([a,b])$ is directed and has the least upper bound x, we write $\bigvee^{\uparrow} A = x$.

The way-below relation $\ll$ is defined in the standard manner: $f \ll g$ if for every directed subset $A \subseteq I_f[a,b]$ with $g \sqsubseteq \vee^{\uparrow} A$ there exists $a \in A$ with $f \sqsubseteq a$.

Proposition 2.6. *For each compact interval $[a,b]$ the functional domain $\mathcal{I}_f([a,b])$ is an effectively given ω-continuous Scott domain.*

Proof. The existance of $\bigvee^{\uparrow} A$ for each directed subset $A \subseteq \mathcal{I}_f([a,b])$ follows from the properties of semicontinuous functions. Indeed, $\bigvee^{\uparrow} A = \left\langle \sup_{f\in A} f^1, \inf_{f\in A} f^2 \right\rangle$, where f^1 is the lower bound and f^2 is the upper bound of f.

Let us prove that $\vee^{\uparrow}(\downarrow f)$ for $f \in \mathcal{I}_f([a,b])$, where $\downarrow f$ denotes the set $\{g \in \mathcal{I}_f([a,b]) | g \ll f\}$. Let U be open and $\mathrm{cl}U = \bar{U} \subset Y_f$. The set $\downarrow f$ contains all functions of the type $g_U^n = \langle a_U^n, c_U^n \rangle$, where

$$a_U^n(x) = \begin{cases} -\infty & \text{if } x \notin U, \\ \inf_{z\in\bar{U}} f^1(z) - \frac{1}{n} & \text{if } x \in U, \end{cases}$$

$$c_U^n(x) = \begin{cases} +\infty & \text{if } x \notin U, \\ \sup_{z\in\bar{U}} f^2(z) + \frac{1}{n} & \text{if } x \in U, \end{cases}$$

By the properties of semicontinuous functions, $\vee^{\uparrow}\{g_U^n | \bar{U} \subset Y_f,\ n \in \omega\} = f$, so $\vee^{\uparrow} \downarrow f = f$.

It is obvious that the function domain $\mathcal{I}_f([a,b])$ is ω-continuous. An example for a countable basis is the set $\mathcal{I}_{f,0}([a,b]) = \{b_n\}_{n\in\omega} \cup \{\perp_{[a,b]}\}$, where the lower bound b_n^1 and the upper bound b_n^2 of b_n satisfy the following conditions: there exist $a = a_0 \ldots \leq a_i \leq \ldots \leq a_n = b$ such that

1. for all $x \in (a_i, a_{i+1})$ $b_n^1(x) = -\infty$ and $b_n^2(x) = +\infty$ or $b_n^1(x) = \alpha_i x + \beta_i$ and $b_n^2(x) = \gamma_i x + \zeta_i$;
2. if for $x \in (a_i, a_{i+1}) \cup (a_{i+1}, a_{i+2})$ b_n^1 and b_n^2 are finite then $b_n^1(a_{i+1}) = \alpha_i a_{i+1} + \beta_i = \alpha_{i+1} a_{i+1} + \beta_{i+1}$ and $b_n^2(a_{i+1}) = \gamma_i a_{i+1} + \zeta_i = \gamma_{i+1} a_{i+1} + \zeta_{i+1}$;
3. if b_n^1 and b_n^2 are infinite on (a_i, a_{i+1}) then $b_n^1(a_i) = b_n^1(a_{i+1}) = -\infty$ and $b_n^2(a_i) = b_n^2(a_{i+1}) = +\infty$, where $a_i,\ \alpha_i,\ \beta_i,\ \gamma_i,\ \zeta_i \in \mathbf{Q}$.

Using the standard numbering of the set of piecewise linear functions with rational coefficients, it is easy to prove that $\mathcal{I}_{f,0}([a,b])$ is countable and effective. □

In the same way we can construct an interval domain $\mathcal{I}_f([a,b]^n)$ for $n \in \omega$.

Corollary 2.7. *For each compact n-cube $[a,b]^n$ the interval domain $\mathcal{I}_f([a,b]^n)$ is an effectively given ω-continuous Scott domain.*

Proof. It is similar to the proof of Proposition 2.6. □

Now we consider a useful property of the way-below relation $\ll$. Thus, $f \ll g$ if and only if these functions are separated.

Definition 2.8. *Let f and g be lower semicontinuous functions, $\mathrm{cl}Y_f \subset Y_g$ and $f \leq g$. The functions f, g are said to be separated if there exists a continuous on Y_g function h such that $f(x) \leq h(x) < g(x)$ for all $x \in Y_g$.*

Let f and g be upper semicontinuous functions, $\mathrm{cl}Y_g \subset Y_f$ and $f \leq g$. The functions f and g are said to be separated if there exists a continuous on Y_f function h such that $f(x) < h(x) \leq g(x)$ for all $x \in Y_f$.

Let $f : \mathbf{R} \to \mathcal{I}$ and $g : \mathbf{R} \to \mathcal{I}$ be continuous. The functions f and g are said to be separated if their lower bounds f^1, g^1 and their upper bounds f^2, g^2 are separated.

Proposition 2.9. *Let f and g be lower semicontinuous and* $\mathrm{cl}Y_f \subset Y_g$, $f \le g$. *The following assertions are equivalent.*

1. f and g are separated;

2. there exists a step upper semicontinuous function h such that $f(x) \le h(x) < g(x)$ for $x \in Y_g$;

3. there exists an upper semicontinuous function h such that $f(x) \le h(x) < g(x)$ for $x \in Y_g$;

4. $(\forall x \in Y_g)\exists U_x(\exists t_x > 0)(\forall z, w \in U_x)(g(z) > f(w) + t_x)$, where U_x denotes some neighbourhood of x.

Proof. We prove nontrivial passages.
$1 \to 2$. It follows from the fact that each continuous function is approximated by a step upper semicontinuous functions (see [3]).
$2 \to 1$. See [31].
$2 \to 3$. Obviously.
$3 \to 4$. Let $h : [a, b] \to \mathbf{R}^+$ be upper semicontinuous and $f(x) \le h(x) < g(x)$ for $x \in Y_g$. For $x \in Y_g$ put $\tau = g(x) - h(x)$ and $\epsilon = \frac{t}{3}$. According to upper semicontinuity of h and lower semicontinuity of g, there exists a neighbourhood U_x of x such that for all $z,\ w \in U_x :\ f(z) \le h(z) < h(x) + \epsilon$ and $g(w) > g(x) - \epsilon$. We have $g(w) > g(x) - \epsilon = h(x) + \frac{2}{3}\tau > h(z) + \frac{\tau}{3} \ge f(z)$. For $t_x = \frac{\tau}{3}$ assertion 4 holds.
$4 \to 2$. Let $\{U_x\}_{x \in \mathrm{cl}Y_f}$ have the following property: for all $z, w \in U_x$ $g(z) > f(w) + t_x$. Since $\mathrm{cl}Y_f$ is compact, we can construct a finite set $\{\bar{U}_{x_i}\}_{i \le n}$ such that:

1. $\bar{U}_{x_i}$ is closed;
2. $\bar{U}_{x_i} \cap \bar{U}_{x_j}$ is one-element or empty;
3. $Y_f \subseteq \bigcup_{i \le n} \bar{U}_{x_i}$.

Put $h(x) = \sup\{y | y \ge f(x) \wedge (\exists i (x \in \bar{U}_{x_i}) \wedge (y \le \inf\limits_{z \in \bar{U}_{x_i} \cap \mathrm{cl}Y_f} g(z) - t_{x_i})\}$.

By the properties of lower semicontinuity of g, the function h is a required one. □

Proposition 2.10. *Let f and g be upper semicontinuous and* $\mathrm{cl}Y_g \subset Y_f$, $f \le g$. *The following assertions are equivalent.*

1. f and g are separated;

2. there exists a step lower semicontinuous function h such that $f(x) < h(x) \le g(x)$ for $x \in Y_f$;

3. there exists a lower semicontinuous function h such that $f(x) < h(x) \le g(x)$ for $x \in Y_f$;

4. $(\forall x \in Y_f)\exists U_x(\exists t_x > 0)(\forall z, w \in U_x)(g(z) > f(w) + t_x)$, where U_x denotes some neighbourhood of x.

Proof. It is similar to the proof of Proposition 2.9. □

Lemma 2.11. *Let A be a directed set of lower semicontinuous functions and* $\lim_{a \in A} a(x) = g(x)$. *For a compact V and some $c \in \mathbf{R}$ the following assertion holds. If $g(x) > c$ for all $x \in V$, then there exists $a \in A$ such that $a(x) > c$ for all $x \in V$.*

Proof. Clearly, for all $x \in V$ there exists $a_x \in A$ such that $a_x(x) > c$. By the definition of lower semicontinuity, there exists a neighbourhood U_x of x with $(\forall z \in U_x)\,(a_x(z) > c)$. The set $\{U_x\}_{x \in V}$ covers the compact V, so we can extract a finite subcovering $\{U_{x_i}\}_{i \le n}$. For all $z \in U_{x_i}$ we have $a_{x_i}(z) > c$. By the definition of a directed set, there exists a function $a \in A$ such that $a(x) > a_{x_i}(x)$ for all $x \in V$. This is a required function. □

Theorem 2.12. *Continuous functions $f : [a,b] \to \mathcal{I}$ and $g : [a,b] \to \mathcal{I}$ are separated if and only if $f \ll g$.*

Proof. Let f and g be separated and f^1, g^1 be their lower bounds. We show that for a directed set $A \subseteq \mathcal{I}_f[a,b]$ with $g \sqsubseteq \vee^{\uparrow} A$ there exists $a \in A$ such that $f \sqsubseteq a$. It is sufficient to prove that there exists a with the lower bound a^1 such that $a^1(x) \geq f^1(x)$ for $x \in [a,b]$. By Proposition 2.9 we have $\mathrm{cl}Y_{f^1} \subseteq Y_{g^1} \wedge (\forall x \in Y_{g^1}) \exists U_x (\exists t_x > 0)(\forall z, w \in U_x)(g^1(z) > f^1(w) + t_x)$, where U_x denotes some neighbourhood of x.

From the set $\{U_x\}_{x \in Y_{g^1}}$ which covers $\mathrm{cl}Y_{f^1}$ we can extract a finite set $\{U_{x_i}\}_{i \leq n}$ such that $\mathrm{cl}Y_{f^1} \subseteq \{U_{x_i}\}_{i \leq n}$. Moreover, it is easy to construct $\{\bar{U}_{x_i}\}_{i \leq m}$ which covers $\mathrm{cl}Y_{f^1}$, where $\bar{U}_{x_i}$ is compact. For $i \leq m$ we define $c_i = \sup\{y | y \leq (\inf_{x \in \bar{U}_{x_i}} g^1(x) - t_{x_i})\}$.

Clearly, $g^1(x) > c_i \geq f^1(x)$ for all $x \in \bar{U}_{x_i}$. From Lemma 2.11 we have that there exists a_i with the lower bound a_i^1 such that $a_i^1(x) > c_i \geq f^1(x)$ for all $x \in \bar{U}_{x_i}$. Since A is directed, there exists $a \sqsupseteq a_i$ for all $i \leq m$. This function is a required one.

Let $f \ll g$. We show that $\mathrm{cl}Y_{f^1} \subseteq Y_{g^1} \wedge (\forall x \in Y_{g^1}) \exists U_x (\exists t_x > 0)(\forall z, w \in U_x)(g^1(z) > f^1(w) + t_x)$, where U_x denotes a neighbourhood of x and f^1, g^1 denote the lower bounds of f and g. For the upper bounds the corresponding assertion is proved by analogy. Obviously, $Y_f \subseteq Y_g$ and $f \sqsubseteq g$. Suppose the contrary. There exists $x \in Y_g$ such that $\forall U_x \forall t_x\, (\exists z, w \in U_x)\, g^1(z) < f^1(w) + t_x$. Let us define $\{U_n\}_{n \in \omega}$ by the following rule:

$$\begin{array}{lcll} U_n & = & (x - \frac{1}{n}, x + \frac{1}{n}) & \text{if } x \in (a,b), \\ U_n & = & [a, a + \frac{1}{n}) & \text{if } x = a, \\ U_n & = & (b - \frac{1}{n}, b] & \text{if } x = b. \end{array}$$

There exists n_0 such that for all $n \geq n_0$ we have $U_n \subseteq Y_g^1$ and there exists $w_n \in U_n$ with $\inf_{z \in \bar{U}_n} g^1(z) < f^1(w_n) + \frac{1}{n}$. We construct an increasing sequence of lower semicontinuous functions such that the limit of this sequence is g^1, but there is no n such that $a_n(y) \geq f^1(y)$ for all $y \in [a,b]$.

Put

$$a_n^1(y) = \begin{cases} g^1(y) & \text{if } y \notin U_n, \\ \inf_{z \in \bar{U}_n} g^1(z) - \frac{1}{n} & \text{if } y \in U_n, \end{cases}$$

where $\bar{U}_n$ is the closure of U_n. It is easy to see that $\lim\limits_{n \to \infty} a_n^1(y) = g^1(y)$ for all $y \in [a,b]$. For $y \neq x$ it is obvious. We consider the nontrivial case when $y = x$. Suppose the contrary: there exists c such that $a_n^1(x) < c < g^1(x)$. By lower semicontinuity of g, there exists N with $\inf_{z \in \bar{U}_N} g^1(z) > c + \frac{1}{N}$. So $\inf_{z \in \bar{U}_N} g^1(z) = a_N^1(x) > c$. This is a contradiction.

On the one hand, $\lim\limits_{n \to \infty} a_n^1(y) = g^1(y)$ for all $y \in [a,b]$ and, on the other hand, there is no n such that $a_n(y) \geq f^1(x)$ for all $y \in [a,b]$ because for all n $a_n^1(w_n) < f^1(w_n)$. This is a contradiction with the assumption $f \ll g$.

Let us prove that $\mathrm{cl}Y_f \subseteq Y_g$. Suppose the contrary. There exists a sequence $\{x_n\}_{n \in \omega}$ such that for all n $x_n \in Y_f \subseteq Y_g$, but $\lim\limits_{x \to \infty} x_n = x \notin Y_g$, i.e. $g(x) = \perp$. We can extract a subsequence $\{x_{m_n}\}_{n \in \omega}$ such that $|x_{m_n} - x| < \frac{1}{n}$. We define a sequence of lower semicontinuous functions in the following way:

$$a^1(x) = \begin{cases} g^1(y) & \text{if } |y - x| > \frac{1}{n}, \\ -\infty & \text{if } |y - x| \leq \frac{1}{n}. \end{cases}$$

On the one hand $\lim\limits_{n \to \infty} a_n^1(y) = g^1(y)$ for all $y \in [a,b]$ and on the other hand $-\infty = a_n^1(x_{m_n}) < f^1(x_{m_n}) \neq -\infty$. This is a contradiction with the assumption $f \ll g$.

□

Now we introduce the notions of computable operators and computable functionals defined on total continuous real-valued functions. Below we use the standard notion of continuity of a total operator $F : \mathcal{I}_f([a,b]) \to \mathcal{I}_f([c,d])$ w.r.t. the Scott-topologies on $\mathcal{I}_f([a,b])$ and $\mathcal{I}_f([c,d])$.

Definition 2.13. *Let $\mathcal{I}_f([a,b])$, $\mathcal{I}_f([c,d])$ be some function domains and $\mathcal{I}_{f,0}([a,b]) = \{b_i\}_{i\in\omega}$, $\mathcal{I}_{f,0}([c,d]) = \{c_i\}_{i\in\omega}$ be their effective bases constructed as in Proposition 2.6. A continuous total operator $F : \mathcal{I}_f([a,b]) \to \mathcal{I}_f([c,d])$ is computable, if the relation $c_m \ll F(b_n)$ is computable enumerable in n and m, where $b_n \in \mathcal{I}_{f,0}([a,b])$ and $c_m \in \mathcal{I}_{f,0}([c,d])$.*

Definition 2.14. *An operator $F : C[a,b] \to C[c,d]$ is computable, if* dom F *is open and there exists a computable operator $F^* : \mathcal{I}_f([a,b]) \to \mathcal{I}_f([c,d])$ such that*

$$F(f) = g \leftrightarrow F^*(\hat{f}) = \hat{g},\ \text{where } \hat{f}(x) = \{f(x)\},\ \hat{g}(x) = \{g(x)\}.$$

Definition 2.15. *A functional $F : C[a,b] \times [c,d] \to \mathbf{R}$ is computable, if there exists a computable operator $F^* : C[a,b] \to C[c,d])$ such that*

$$F(f,x) = y \leftrightarrow F^*(f)(x) = y.$$

Proposition 2.16. *Computable operators and functionals defined on continuous real-valued functions are continuous w.r.t. the standard topology induced by the uniform norm.*

Proof. It follows from the definition and continuity of corresponding operator
$F^* : \mathcal{I}_f([a,b]) \to \mathcal{I}_f([c,d])$. □

To introduce computability of a functional of the type $F : C[a,b] \times \mathbf{R} \to \mathbf{R}$, we use an effective sequence of domains $\{\mathcal{I}_f([-n,n])\}_{n\in\omega}$ with conforming bases in the following sense. We consider a sequence of bases $\{\mathcal{I}_{f,0}([-n,n])\}_{n\in\omega} = \{\{b_i^n\}_{i\in\omega}\}_{n\in\omega}$ with the homomorphisms $\mathrm{res}_{m,n} : \mathcal{I}_f([-m,m]) \to \mathcal{I}_f([-n,n])$ of restrictions for $m > n$ defined by the natural rules $\mathrm{res}_{m,n}(b_i^m) = b_i^m|_{[-n,n]} = b_i^n$ and $\mathrm{res}_{m,n}(\bot_{[-m,m]}) = \bot_{[-n,n]}$.

Definition 2.17. *A sequence $\{F_k\}_{k\in\omega}$ of computable operators of the type*
$F_k : \mathcal{I}_f[a,b] \to \mathcal{I}_f[-n,n]$ is uniformly computable if $\{\langle k,n,m\rangle \,|F_k(b_n^k) \gg b_m^k\}$ is recursively enumerable in k, n and m.

Definition 2.18. *A sequence $\{F_k\}_{k\in\omega}$ of computable operators of the type*
$F_k : C[a,b] \to C[-n,n]$ is uniformly computable, if there exists a uniformly computable sequence of computable operators $\{F_k^*\}_{k\in\omega}$ of the type $F_k : \mathcal{I}_f[a,b] \to \mathcal{I}_f[-n,n]$ such that

$$F_k(f) = g \leftrightarrow F_k^*(\hat{f}) = \hat{g},\ \text{where } \hat{f}(x) = \{f(x)\},\ \hat{g}(x) = \{g(x)\},\ k \in \omega.$$

Definition 2.19. *A functional $F : C[a,b] \times \mathbf{R} \to \mathbf{R}$ is computable, if there exists a uniformly computable sequence $\{F_k^*\}_{k\in\omega}$ of computable operators of the types $F_k^* : C[a,b] \to C[-k,k]$ such that*

$$F(f,x) = y \leftrightarrow \forall k\, (x \in [-k,k]) \to (F_k^*(f)(x) = y)\,.$$

Note that for $m > n$ the condition $\mathrm{res}_{m,n}(F_m^*(f)) = F_n^*(f)$ holds by construction.

Proposition 2.20. *A computable functional $F : C[a,b] \times \mathbf{R} \to \mathbf{R}$ is continuous w.r.t. the standard topology induced by the uniform norm.*

Proof. It follows from the definition and continuity of the corresponding operators $F_k^* : \mathcal{I}_f([a,b]) \to \mathcal{I}_f([-k,k])$ for $k \in \omega$. □

In the same way we can define computability of functionals of the type $F : C[a,b] \times \mathbf{R}^n \to \mathbf{R}$.

Corollary 2.21. *A computable functional $F : C[a,b] \times \mathbf{R}^n \to \mathbf{R}$ is continuous.*

Proof. It follows from the definition and continuity of the corresponding operators $F_k^* : \mathcal{I}_f([a,b]) \to \mathcal{I}_f([-k,k]^n)$ for $k \in \omega$. □

3. Definability of computable functions and functionals

To semantically characterise computable real-valued functions, operators and functionals via validity of finite formulas, we use comparative analyses with real-valued majorant-computable functions proposed in [14, 15] and generalised computable operators and functionals introduced below.

The computation of real-valued function is an infinite process that produces approximations closer and closer to the result. The class of majorant-computable real-valued functions has clear and exact classifications in logical and topological terms.

3.1. Majorant-computable function and generalised computable operators and functionals

To recall the notion of majorant-computability and to introduce generalised computability, let us construct the set of hereditarily finite sets $\mathrm{HF}(M)$ over a model $\mathbf{M}$. This structure is rather well studied in the theory of admissible sets [2] and permits us to define the natural numbers, to code and store information via formulas.

Let $\mathbf{M}$ be a model whose language σ_0 contains no function symbols and whose carrier set is M. We construct the set of hereditarily finite sets, $\mathrm{HF}(M)$, as follows:

1. $\mathrm{S}_0(M) \rightleftharpoons M$, $\mathrm{S}_{\mathrm{n}+1}(M) \rightleftharpoons \mathcal{P}_\omega(\mathrm{S}_\mathrm{n}(M)) \cup \mathrm{S}_\mathrm{n}(M)$, where $n \in \omega$ and for every set B, $\mathcal{P}_\omega(B)$ is the set of all finite subsets of B.
2. $\mathrm{HF}(M) = \bigcup_{n\in\omega} \mathrm{S}_n(M)$.

We define $\mathbf{HF}(\mathbf{M})$ as the following model:

$$\mathbf{HF}(\mathbf{M}) \rightleftharpoons \left\langle \mathrm{HF}(M), M, \sigma_0, \emptyset_{\mathbf{HF}(\mathbf{M})}, \in_{\mathbf{HF}(\mathbf{M})} \right\rangle,$$

where $\emptyset_{\mathbf{HF}(\mathbf{M})}$ and the binary predicate symbol $\in_{\mathbf{HF}(\mathbf{M})}$ has the set-theoretic interpretation. Below we will use the notations $\in$ and $\emptyset$. Denote $\sigma = \sigma_0 \cup \{\in, \emptyset\}$. The notions of a term and an atomic formula are given in a standard manner.

The set of Δ_0-*formulas* is the closure of the set of atomic formulas in the language σ under $\wedge, \vee, \neg, (\exists x \in t)$ and $(\forall x \in t)$, where $(\exists x \in t)\ \varphi$ denotes $\exists x(x \in t \wedge \varphi)$ and $(\forall x \in t)\ \varphi$ denotes $\forall x(x \in t \to \varphi)$. The set of Σ-*formulas* is the closure of the set of Δ_0 formulas under $\wedge, \vee, (\exists x \in t), (\forall x \in t)$, and $\exists$. We define Π-*formulas* as negations of Σ-formulas.

Definition 3.1. *1.A set $B \subseteq \mathrm{HF}(M)$ is Σ-definable, if there exists a Σ-formula $\Phi(x)$ such that $x \in B \leftrightarrow \mathbf{HF}(\mathbf{M}) \models \Phi(x)$.*

2.A function $f : \mathrm{HF}(M) \to \mathrm{HF}(M)$ is Σ-definable, if there exists a Σ-formula $\Phi(x, y)$ such that $f(x) = y \leftrightarrow \mathbf{HF}(\mathbf{M}) \models \Phi(x, y)$.

In a similar way, we define the notions of Π-*definable functions* and *sets*. The class of Δ-*definable functions* (*sets*) is the intersection of the class of Σ-definable functions (sets) and the class of Π-definable functions (sets).

Note that the sets M and M^n are Δ_0–definable. This fact makes $\mathbf{HF}(\mathbf{M})$ a suitable domain for studying functions from M^k to M. Below, when we say about definability, we mean definability in $\mathbf{HF}(\mathbf{R})$. To introduce the definition of majorant-computability, we use a class of Σ-, Π-definable sets as the basic classes. So, we recall some usefull properties of Σ-, Π-definable subsets of $\mathbf{R}^n$.

Proposition 3.2. *Let $\mathbf{R}$ be the reals with the language $\sigma_0 = \langle 0, 1, +, \cdot, < \rangle$.*

1.The set $\mathbf{HF}(\emptyset)$ and the predicate of equality on $\mathbf{HF}(\emptyset)$ are Σ-definable.

2.The set $\{\langle n, r\rangle \mid n$ is a Gödel number of a Σ-formula Φ, $r \in \mathbf{R}$, and $\mathbf{HF}(\mathbf{R}) \models \Phi(x)\}$ is Σ-definable.

3.A set $B \subseteq \mathbf{R}^n$ is Σ-definable if and only if there exists an effective sequence of formulas in the language σ_0 with existential quantifiers over the reals, $\{\Phi_s(x)\}_{s\in\omega}$, such that $x \in B \leftrightarrow \mathbf{R} \models \bigvee_{s\in\omega} \Phi_s(x)$.

4.A set $B \subseteq \mathbf{R}^n$ is Π-definable if and only if there exists an effective sequence of formulas in the language σ_0 with universal quantifiers over the reals, $\{\Phi_s(x)\}_{s\in\omega}$, such that $x \in B \leftrightarrow \mathbf{R} \models \bigwedge_{s\in\omega} \Phi_s(x)$.

Proof. The claim immediately follows from the properties of the set of hereditarily finite sets (see [8, 15]). □

Let us recall the notion of majorant-computability for real-valued functions proposed and investigated in [14, 15]. We use the class of Σ- and Π-definable sets as the basic classes. A real-valued function is said to be *majorant-computable* if we can construct a special kind of nonterminating process computing approximations closer and closer to the result.

Definition 3.3. *A function $f : \mathbf{R}^n \to \mathbf{R}$ is called majorant-computable if there exists an effective sequence of Σ-formulas $\{\Phi_s(\mathbf{x}, y)\}_{s\in\omega}$ and an effective sequence of Π-formulas $\{G_s(\mathbf{x}, y)\}_{s\in\omega}$ such that the following conditions hold.*

1.For all $s \in \omega$, $\mathbf{x} \in \mathbf{R}^n$, the formulas Φ_s and G_s define nonempty intervals $< \alpha_s, \beta_s >$ and $< \delta_s, \gamma_s >$.

2.For all $\mathbf{x} \in \mathbf{R}^n$, the sequences $\{< \alpha_s, \beta_s >\}_{s\in\omega}$ and $\{< \delta_s, \gamma_s >\}_{s\in\omega}$ decrease monotonically and $< \alpha_s, \beta_s >\subseteq< \delta_s, \gamma_s >$ for all $s \in \omega$.

3.For all $\mathbf{x} \in \mathrm{dom}(f)$, $f(\mathbf{x}) = y \leftrightarrow \bigcap_{s\in\omega} < \alpha_s, \beta_s >= \{y\}$ and $\bigcap_{s\in\omega} < \delta_s, \gamma_s >= \{y\}$ holds.

The sequence $\{F_s\}_{s\in\omega}$ in Definition 3.3 is called a *sequence of Σ-approximations* for f. The sequence $\{G_s\}_{s\in\omega}$ is called a *sequence of Π-approximations* for f. As we can see, the process which carries out the computation is represented by two effective procedures. These procedures produce Σ-formulas and Π-formulas which define approximations closer and closer to the result.

The following theorem connects a majorant-computable function with validity of finite formulas in the set of hereditarily finite sets, $\mathbf{HF}(\mathbf{R})$.

Proposition 3.4. *For all functions $f : \mathbf{R}^n \to \mathbf{R}$ the following assertions are equivalent:*

1.The function f is majorant-computable.

2.There exist Σ–formulas $A(\mathbf{x}, y)$, $B(\mathbf{x}, y)$ such that $A(\mathbf{x}, \cdot) < B(\mathbf{x}, \cdot)$ and

$$f(\mathbf{x}) = y \leftrightarrow (A(\mathbf{x}, \cdot) < y < B(\mathbf{x}, \cdot) \wedge \\ \{z \mid A(\mathbf{x}, z)\} \cup \{z \mid B(\mathbf{x}, z)\} = \mathbf{R} \setminus \{y\}).$$

Proof. $\to$)Let $f : \mathbf{R}^n \to \mathbf{R}$ be majorant-computable. By Definition 3.3 , there exist a sequence $\{F_s\}_{s\in\omega}$ of Σ-approximations for f and a sequence $\{G_s\}_{s\in\omega}$ of Π-approximations for f. Put

$$A(\mathbf{x}, y) \rightleftharpoons (\exists s \in \omega)\,(y \notin< \delta_s, \gamma_s > \wedge\, (\exists z \in< \alpha_s, \beta_s >)\,(y < z))$$

and

$$B(\mathbf{x}, y) \rightleftharpoons (\exists s \in \omega)\,(y \notin< \delta_s, \gamma_s > \wedge\, (\exists z \in< \alpha_s, \beta_s >)\,(y > z))\,.$$

By construction, A and B are the sought formulas.
$\leftarrow$) Let A and B satisfy the requirements of the theorem. Let us construct approximations in the following way.

$$F_s(\mathbf{x},y) \rightleftharpoons \exists z \exists v\, (A(\mathbf{x},z) \wedge B(\mathbf{x},v) \wedge y \in (z,v) \wedge v - z < 1/s)\,,$$
$$G_s(\mathbf{x},y) \rightleftharpoons \forall z\, (A(\mathbf{x},z) \to z - y < 1/s) \wedge \forall z\, (B(\mathbf{x},z) \to y - z < 1/s)\,.$$

□

As a corollary we note that a total real-valued function is majorant-computable if and only if its epigraph and ordinate set are Σ-definable (i.e. effective sets). The same proposition holds for a total function $f : [a,b]^n \to \mathbf{R}$ for some compact n-cube $[a,b]^n$.

Definition 3.5. *A real-valued function f is said to be shared by Σ-formulas φ_1, φ_2 if*

$$f|_{[x_1,x_2]} > c \leftrightarrow \mathbf{HF(R)} \models \varphi_1(x_1,x_2,c),$$
$$f|_{[x_1,x_2]} < c \leftrightarrow \mathbf{HF(R)} \models \varphi_2(x_1,x_2,c).$$

Proposition 3.6. *A real-valued function is majorant-computable if and only if it is shared by two Σ-formulas.*

Proof. The claim immediate follows from Proposition 3.4. □

Theorem 3.7. *The class of computable real-valued functions coincides with the class of majorant-computable real-valued functions.*

Proof. Without loss of generality we consider a function $f : \mathbf{R} \to [0,1]$. Let $f^* : \mathcal{I} \to \mathcal{I}_{[0,1]}$ be computable and $f^*(\{x\}) = \{f(x)\}$. For $n \in \omega$, we define $A_n = \{x \in \mathbf{R} \mid \mu(f^*(\{x\})) < \frac{1}{n}\}$, where μ is the natural measure defined on $\mathcal{I}_{[0,1]}$. It is easy to see that A_n is a Σ-definable open set, and $\mathrm{dom}(f) = \bigcap_{n\in\omega} A_n$.
Because each Σ-definable subset of $\mathbf{R}$ is an effective union of open intervals, we can denote $A_1 = \bigcup_{i\in\omega}(\alpha_i,\beta_i)$, where α_i, $\beta_i \in \mathbf{Q}$ and $\alpha_i \le \beta_i$.

The following formulas satisfy the conditions of Proposition 3.4 :

$$A(x,z) \rightleftharpoons x \in A_1 \wedge (\exists a \in \mathbf{Q})\, (\exists b \in \mathbf{Q})\, (\exists y \in \mathbf{Q})\, (x \in (a,b) \wedge y > z \wedge \\ [y,y+1] \ll f^*([a,b]),$$

$$B(x,z) \rightleftharpoons x \in A_1 \wedge (\exists a \in \mathbf{Q})\, (\exists b \in \mathbf{Q})\, (\exists y \in \mathbf{Q})\, (x \in (a,b) \wedge y < z \wedge \\ [y,y+1] \ll f^*([a,b])$$

By Proposition 2.13, f is majorant-computable.

Let f be majorant-computable and A and B satisfy the properties from Proposition 3.4. We construct a computable function $f^* : \mathcal{I} \to \mathcal{I}$ such that $f^*(\{x\}) = \{f(x)\}$.

Put $f^*([a,b]) = \bigcup_{x\in[a,b]} f^{**}(x)$, where the auxiliary function f^{**} is defined in the following way:

$$f^*([a,b]) = \begin{cases} \cap\{[u,v] \mid u,v \in \mathbf{Q},\ <x,u> \in A,\ <x,v> \in B\} & \text{if such } u \text{ and } v \text{ exist} \\ \perp & \text{otherwise} \end{cases}$$

It is easy to see that f is continuous and the set $E = < a,b,c,d > \mid a,b,c,d \in \mathbf{Q},\ [c,d] \ll f^*([a,b])$ is Σ-definable by the following Σ-formula

$$\exists x \in (a,b)\, (< x,c > \in A \wedge < x,d > \in B)\,.$$

So the function f is computable.

□

To introduce generalised computability of operators and functionals, we extend the language σ by two 3-ary predicates U_1 and U_2.

Definition 3.8. *A total operator $F^* : \mathcal{I}_f[a,b] \to \mathcal{I}_f[c,d]$ is said to be shared by two Σ-formulas φ_1 and φ_2 if the following assertions hold. If $F^*(\langle u^1, u^2\rangle) = \langle h^1, h^2\rangle$ then*

$$h^1|_{[x_1,x_2]} > z \leftrightarrow \mathbf{HF}(\mathbf{R}) \models \varphi_1(U_1, U_2, x_1, x_2, z);$$
$$h^2|_{[x_1,x_2]} < z \leftrightarrow \mathbf{HF}(\mathbf{R}) \models \varphi_2(U_1, U_2, x_1, x_2, z),$$

where $U_1(x_1,x_2,c) \rightleftharpoons u^1|_{[x_1,x_2]} > c, U_2(x_1,x_2,c) \rightleftharpoons u^2|_{[x_1,x_2]} < c$ and the predicates U_1 and U_2 positively occur in φ_1, φ_2.

Definition 3.9. *An operator $F : C[a,b] \to C[c,d]$ is said to be generalised computable, if there exists an operator $F^* : \mathcal{I}_f[a,b] \to \mathcal{I}_f[c,d]$ which is shared by two Σ-formulas and $F(f) = F^*(\hat{f})$, where $\hat{f}(x) = \{f(x)\}$.*

Definition 3.10. *A functional $F : C[a,b] \times [c,d] \to \mathbf{R}$ is said to be generalised computable, if there exists a computable operator $F^* : C[a,b] \to C[c,d]$ such that $F(f,x) = F^*(f)(x)$.*

Definition 3.11. *A functional $F : C[a,b] \times \mathbf{R} \to \mathbf{R}$ is said to be generalised computable, if there exists an effective sequence of computable operators $\{F_n^*\}_{n\in\omega}$ of the types $F^* : C[a,b] \to C[-n,n]$ such that*

$$F(f,x) = y \leftrightarrow \forall n\,(-n \le x \le n \to F_n^*(f)(x))\,.$$

Theorem 3.12. *An operator $F : C[a,b] \to C[c,d]$ is computable if and only if it is generalised computable.*

Proof. Let $F : C[a,b] \to C[c,d]$ be computable. To show generalised computability of its corresponding operator $F^* : \mathcal{I}_f[a,b] \to \mathcal{I}_f[c,d]$, we construct two Σ-formulas φ_1, φ_2 satisfying the conditions of Definition 17. Let $\mathcal{I}_{f,0}([a,b]) = \{b_i\}_{i\in\omega}$ and $\mathcal{I}_{f,0}([c,d]) = \{c_i\}_{i\in\omega}$ be effective bases constructed as in Proposition 2.6 for $\mathcal{I}_f([a,b])$ and $\mathcal{I}_{f,0}([c,d])$.

Suppose $F^*(u) = h$. By Proposition 3.4 and Corollary 3.6 the relation $b_n \ll u$ is definable by Σ-formulas with positive occurrences of U_1 and U_2, where $U_1(r_1,r_2,c) \rightleftharpoons u^1|_{[r_1,r_2]} > c$, $U_2(r_1,r_2,c) \rightleftharpoons u^2|_{[r_1,r_2]} < c$. Therefore the set $\{(n,m) | u \gg c_n \wedge F^*(b_n) \gg b_m\}$ is definable by some Σ-formula $\Phi(n,m,U_1,U_2)$. Then $F^*(u) \gg c_m \leftrightarrow \mathbf{HF}(\mathbf{R}) \models \exists n \Phi(n,m,U_1,U_2)$.

Put

$$\varphi_1(U_1,U_2,x_1,x_2,z) \rightleftharpoons \exists m \exists n \left(b_m^1|_{[x_1,x_2]} > z\right) \wedge \Phi(n,m,U_1,U_2),$$

$$\varphi_2(U_1,U_2,x_1,x_2,z) \rightleftharpoons \exists m \exists n \left(b_m^2|_{[x_1,x_2]} < z\right) \wedge \Phi(n,m,U_1,U_2).$$

Clearly, φ_1, φ_2 are required formulas.

Let $F : C[a,b] \to C[c,d]$ be generalised computable. We prove computability of its corresponding operator $F^* : \mathcal{I}_f[a,b] \to \mathcal{I}_f[c,d]$. Monotonicity of F^* follows from positive occurrences of U_1 and U_2 in the formulas φ_1 and φ_2.

Because $\mathcal{I}_f[a,b]$ and $\mathcal{I}_f[c,d]$ are ω-continuous domains, it is enough to prove that F^* preserves suprema of a countable directed set.

Let $A = \{< u_n^1, u_n^2 >\}_{n\in\omega}$ and $\bigvee^{\uparrow} A =< u^1, u^2 >$. Put $U_{1n}(x_1,x_2,c) \rightleftharpoons u_n^1|_{[x_1,x_2]} > c$ and $U_{2n}(x_1,x_2,c) \rightleftharpoons u_n^2|_{[x_1,x_2]} < c$ for $n \in \omega$ and $U_1(x_1,x_2,c) \rightleftharpoons u^1|_{[x_1,x_2]} > c$, $U_2(x_1,x_2,c) \rightleftharpoons u^2|_{[x_1,x_2]} < c$.

By Lemma 11, if $u^1|_{[x_1,x_2]} > c$ then there exists n such that $u_n^1|_{[x_1,x_2]} > c$, and if $u^2|_{[x_1,x_2]} > c$ then there exists n such that $u_n^2|_{[x_1,x_2]} < c$.

So $U_1(x_1,x_2,c) = \bigvee_{n\in\omega} U_{1n}(x_1,x_2,c)$ and $U_2(x_1,x_2,c) = \bigvee_{n\in\omega} U_{2n}(x_1,x_2,c)$.

By the properties of Σ-formulas and positive occurrences of U_1 and U_2 in φ_1 and φ_2,

$$\varphi_1(U_1,U_2,x_1,x_2,c) \leftrightarrow \bigvee_{n\in\omega} \varphi_{1n}(U_1,U_2,x_1,x_2,c),$$
$$\varphi_2(U_1,U_2,x_1,x_2,c) \leftrightarrow \bigvee_{n\in\omega} \varphi_{2n}(U_1,U_2,x_1,x_2,c).$$

Hence it is clear that $F^*(\bigvee^{\uparrow} A) = \bigvee^{\uparrow} F^*(A)$.

Now we show that the set $\{(n,m)|F^*(b_n) \gg c_m\}$ is Σ-definable and, as a consequence, is computable enumerable in n and m. Let $F^*(< b_n^1, b_n^2 >) =< h^1, h^2 >$. Since b_n^1, b_n^2, c_m^1 and c_m^2 are piecewise linear, it is obvious that the sets $b_n^1|_{[x_1,x_2]} > c$, $b_n^2|_{[x_1,x_2]} < c$ and $c_m^1|_{[x_1,x_2]} > c$, $c_m^2|_{[x_1,x_2]} < c$ are Σ-definable. As is evident from the definition of F^*, the sets $h^1|_{[x_1,x_2]} > c$, $h^2|_{[x_1,x_2]} < c$ are Σ-definable too. By Proposition 2.9, there exist step upper semicontinuous functions s^1 and s^2 such that $c_m^1(x) < s^1(x) < h^1(x)$ and $c_m^2(x) > s^2(x) > h^2(x)$ for $x \in [c,d]$.

As one can see, the following Σ-formula

$$\exists x_0 \dots \exists x_n \exists y_1 \dots \exists y_n \exists z_1 \dots \exists z_n \bigwedge_{i\le n} \Big((c_m^1|_{[x_i,x_{i+1}]} < y_i\Big) \wedge \Big(h^1|_{[x_i,x_{i+1}]} > y_i\Big) \wedge$$
$$\Big(c_m^2|_{[x_i,x_{i+1}]} > z_i\Big) \wedge \Big(h^2|_{[x_i,x_{i+1}]} < z_i\Big)$$

defines the set $\{(n,m)|F^*(b_n) \gg c_m\}$. As a consequence this set is computable enumerable in n and m. □

Note that using the previous theorem one can elegantly prove computability of such functions as $\sup_{x\in[x_1,x_2]} f(x)$, $\inf_{x\in[x_1,x_2]} f(x)$ and Riemann integral on $[x_1,x_2]$.

Corollary 3.13. *A functional $F : C[a,b] \times [c,d] \to \mathbf{R}$ is computable if and only if it is generalised computable.*

Proof. The claim follows from generalised computability of its corresponding operators. □

Corollary 3.14. *A functional $F : C[a,b] \times \mathbf{R} \to \mathbf{R}$ is computable if and only if it is generalised computable.*

Proof. The claim follows from the property of Σ- formulas: an effective sequence of Σ-formulas is equivalent to a Σ-formula. □

3.2. Semantic characterisation of computable functions and functionals

After the mentions of the main properties of majorant-computable real-valued functions and generalised computable operators and real-valued functionals, we pass to computable ones.

Corollary 3.15. *For a function $f : \mathbf{R}^n \to \mathbf{R}$ the following assertions are equivalent:.*

1. The function f is computable.

2. There exist Σ–formulas $A(\mathbf{x},y)$ and $B(\mathbf{x},y)$ such that $A(\mathbf{x},\cdot) < B(\mathbf{x},\cdot)$ and

$$f(\mathbf{x}) = y \leftrightarrow (A(\mathbf{x},\cdot) < y < B(\mathbf{x},\cdot) \wedge \{z \mid A(\mathbf{x},z)\} \cup \{z \mid B(\mathbf{x},z)\} = \mathbf{R} \setminus \{y\}).$$

Proof. The claim follows from Proposition 3.4 and Theorem 3.7. □

Corollary 3.16. *A real-valued function is computable if and only if it is shared by two Σ-formulas.*

Proof. The claim follows from Proposition 3.6 and Theorem 3.7. □

Proposition 3.17. *Let f be a computable function such that $[a,b] \subseteq \mathrm{dom} f$ and g be a computable function such that $[b,c] \subseteq \mathrm{dom} g$ and $f(b) = g(b)$. Then the function $h(x) = \begin{cases} f(x) & \text{if } x \le b, \\ g(x) & \text{if } x \ge b \end{cases}$ is computable.*

Proof. From Theorem 26 in [6] (cf. [16]) it follows that there exists an effective modulus of continuity w_f for f and an effective modulus of continuity w_g for g. In other words, for every $s \in \omega$ for all $x_1, x_2 \in [a,b]$ and $x_3, x_4 \in [c,d]$ we have

$$\begin{aligned} |x_1 - x_2| \ &< w_f(\frac{1}{s}) \to |f(x_1) - f(x_2)| < \frac{1}{s} \quad \text{and} \\ |x_3 - x_4| \ &< w_g(\frac{1}{s}) \to |g(x_3) - g(x_4)| < \frac{1}{s}. \end{aligned}$$

Put $w_h(\epsilon) = \min\{w_f(\epsilon), w_g(\epsilon)\}$. The following Σ-formula defines the epigraph of the function h.

$$\begin{aligned} &y > h(x) \leftrightarrow (x < b \wedge y > f(x)) \vee (x > b \wedge y > g(x)) \vee \\ &([\exists \epsilon \in \mathbf{Q}^+] \, (|x - b| < w_h(\epsilon) \wedge ([\exists t < b] \, |x - t| < w_h(\epsilon) \wedge y > f(t) + \epsilon)) \end{aligned}$$

Analogously, the ordinate set of h is Σ–definable. By Corollary 3.16, the function h is computable. □

Corollary 3.18. *A functional $F : C[a,b] \times [c,d] \to \mathbf{R}$ is computable if and only if there exists an operator $F^* : \mathcal{I}_f[a,b] \to \mathcal{I}_f[c,d]$ which is shared by two Σ-formulas and $F(f,x) = y \leftrightarrow F^*(\hat{f})(x) = \{y\}$, where $\hat{f}(x) = \{f(x)\}$.*

Proof. It follows from Theorem 3.12. □

Corollary 3.19. *If a computable operator $F : C[a,b] \to C[c,d]$ is defined in a computable function f, then the function $F(f)$ is computable.*

Proof. We only note that if a function u is computable, then the following relations $u^1|_{[x_1,x_2]} > z$ and $u^2|_{[x_1,x_2]} < z$ are Σ-definable. This follows from Proposition 3.6. □

Corollary 3.20. *A total computable operator $F : C[a,b] \to C[c,d]$ maps computable functions to computable functions.*

Proof. It follows from Corollary 3.19. □

Corollary 3.21. *The composition of computable operators is computable.*

Proof. The claim follows from the properties of Σ-formulas and Theorem 3.12. □

Now we introduce a useful recursion scheme which permits us to describe the behaviour of complex systems such as hybrid systems.

Let $\mathcal{F} : C[a,b] \times C[0,1] \times \mathbf{R} \to \mathbf{R}$ and $G : C[a,b] \times [0,1] \to \mathbf{R}$ be computable functionals. Then $F : C[a,b] \times [0,+\infty) \to \mathbf{R}$ is defined by the following scheme:

$$\begin{cases} F(f,t)|_{t\in[0,1]} = G(f,t), \\ F(f,t)|_{t\in(n,n+1]} = \mathcal{F}(f,t,F(f,y+n-1)). \end{cases}$$

Proposition 3.22. *If F is continuous then F is computable, with F defined above.*

Proof. We prove that there exists an effective sequence of generalised computable operators $F_n^* : C[a,b] \to C[0,n]$. For this we state that for each k there exist two Σ-formulas τ_1 and τ_2 which share F_k^*. Clearly, on the m-th step of computation via the recursion scheme, we obtain a computable functional where t ranges over the interval $[m, m+1]$. Hence, there exist two effective sequences of Σ-formulas $\{\tau_1^m\}_{m\in\omega}$ and $\{\tau_2^m\}_{m\in\omega}$ such that for $m \le x_1 \le x_2 \le m+1$ and $F^*(< u^1, u^2 >) =< h^1, h^2 >$ we have

$$h^1|_{[}x_1, x_2] > c \leftrightarrow \tau_1^m(U_1, U_2, x_1, x_2, c),$$
$$h^2|_{[}x_1, x_2] < c \leftrightarrow \tau_2^m(U_1, U_2, x_1, x_2, c),$$

where $U_1(x_1, x_2, c) \rightleftharpoons u^1|_{[x_1,x_2]} > c$, $U_2(x_1, x_2, c) \rightleftharpoons u^2|_{[x_1,x_2]} < c$ and the predicate U_1 and U_2 positively occur in φ_1 and φ_2. The required formula τ_1 can be defined as follows:

$$\tau_1(U_1, U_2, x_1, x_2, c) \rightleftharpoons (\exists i, j \in \mathbf{N}(i < x_1 < i+1)) \wedge (j < x_2 < j+1) \wedge$$
$$(\tau_n^i(U_1, U_2, x_1, i+1, c) \wedge \bigwedge_{i+1 \le m \le j-1} \tau_1^m(U_1, U_2, m, m+1, c) \wedge \tau_1^j(U_1, U_2, j, x_2, c)) \vee$$
$$(\exists j \in \mathbf{N}(j < x_2 < j+1) \wedge \bigwedge_{0 \le m \le j-1} \tau_1^m(U_1, U_2, m, m+1, c) \wedge$$
$$\tau_1^j(U_1, U_2, j, x_2, c)) \vee (\exists i \in \mathbf{N}(i < x_1 < i+1) \wedge \tau_1^i(U_1, U_2, i, x_1, c) \wedge$$
$$\bigwedge_{i \le m \le n-1} \tau_1^m(U_1, U_2, m, m+1, c) \vee \bigwedge_{0 \le m \le n-1} \tau_1^m(U_1, U_2, m, m+1, c).$$

The required formula τ^2 can be defined in the similar way. □

We would like to note that the recursion scheme is a useful tool for formalisation of hybrid systems. Indeed, in this framework the trajectories of the continuous component of hybrid systems (the performance specifications) can be represented by computable functionals which can be constructed by the specifications **SHS** of hybrid systems proposed in [17].

Also we pay attention to the following property. Every continuous total operator $F : C[a, b] \to C[a, b]$ has a continuous extension to the functional domain. This means that there is a continuous operator $F^* : \mathcal{I}_f([a, b]) \to \mathcal{I}_f([a, b])$ such that

$$F(f) = g \leftrightarrow F^*(\hat{f}) = \hat{g}, \text{ where } \hat{f}(x) = \{f(x)\},\ \hat{g}(x) = \{g(x)\}.$$

To prove this fact, we will use the following notion.

Definition 3.23. *Let f be a lower semicontinuous function defined on $[a, b]$ and g be an upper continuous function defined on $[a, b]$. A sequence $\{h_s\}_{s\in\omega}$ of continuous functions defined on $[a, b]$ is said to be closely approximating to $\langle f, g\rangle \in I_f([a, b])$ if*

$$\forall \varepsilon > 0 \exists N \forall n \ge N \left(h_n \in \langle f - \varepsilon, g + \varepsilon\rangle\right).$$

Theorem 3.24. *Every continuous total operator $F : C[a, b] \to C[a, b]$ has a continuous extension to the functional domain.*

Proof. It is enough to define the operator $F^* : I_f^0([a, b]) \to I_f([a, b])$, where $I_f^0([a, b])$ denotes the set $\{h \in (I)_f([a, b]) | h : [a, b] \to I \setminus \perp\}$ which is an ω-continuous Scott domain. Indeed, the operator F^* can be extended to $F^{**} : I_f([a, b]) \to I_f([a, b])$ by the rule:

$$F^{**}(h) = \begin{cases} F^*(h) & \text{if } h \in I_f^0([a, b]), \\ \perp_{[a,b]} & \textit{otherwise}, \end{cases}$$

Note that the set $I_{f,0}^0([a, b]) = \{\langle f, g\rangle \,| f, g \in C[a, b]\}$ can be considered as a basis for $I_f^0([a, b])$.

Let us denote $U^-_{F(f)} = \{(x, t) | F(f)(x) > t\}$ and $U^+_{F(f)} = \{(x, t) | F(f)(x) < t\}$ for a continuous function f.

We first define an auxiliary operator $\mathcal{F}$ defined on the set $I_{f,0}^0([a, b])$ of strips with continuous bounds and then extend it to an operator defined on $I_f^0([a, b])$.

For $\langle f^1, f^2\rangle \in I_f([a, b])$, where f^1 and f^2 are continuous, we define two open sets $U^- < U^+$ by the following rules.
We define $(x, t) \in U^-$ if and only if there exists $\varepsilon > 0$ such that for each sequence $\{h_n\}_{n\in\omega}$ which is closely approximating to $\langle f^1, f^2\rangle$ we have:

$$\exists N\,(\forall n \geq N)\, B((x,t),\varepsilon) \subset U^-_{F(h_n)},$$

where $B((x,t),\varepsilon)$ is the ball of the radius ε centered at (x,t).
By analogy, $(x,t) \in U^+$ if and only if there exists $\varepsilon > 0$ such that for each sequence $\{h_n\}_{n\in\omega}$ which is closely approximating to $\langle f^1, f^2\rangle$ we have:

$$\exists N\,(\forall n \geq N)\, B((x,t),\varepsilon) \subset U^+_{F(h_n)},$$

where $B((x,t),\varepsilon)$ is the ball of the radius ε centered at (x,t).

Let us define $g^1(x) = \sup U^-(x)$ and $g^2(x) = \inf U^+(x)$.

Put $\mathcal{F}(\langle f^1, f^2\rangle) = \langle g^1, g^2\rangle$. Since g^1 is lower semicontinuous and g^2 is upper semicontinuous, the operator $\mathcal{F}$ is well-defined. For $\mathcal{F}$ we denote U^- as $U^-_{\mathcal{F}(\langle f^1,f^2\rangle)}$ and U^+ as $U^+_{\mathcal{F}(\langle f^1,f^2\rangle)}$.

We show that $\mathcal{F}(\langle f, f\rangle) = \langle F(f), F(f)\rangle$. Indeed, a sequence $\{h_n\}_{n\in\omega}$, which is closely approximating to $\langle f, f\rangle$, uniformly converges to f. By continuity of the operator F, the sequence $\{F(h_n)\}_{n\in\omega}$ uniformly converges to $F(f)$. So $U^-_{\mathcal{F}(\langle f,f\rangle)} = U^-_{F(f)}$ and $U^+_{\mathcal{F}(\langle f,f\rangle)} = U^+_{F(f)}$.

Monotonicity of the operator $\mathcal{F}$ follows from the definitions of $U^-_{\mathcal{F}(\langle f^1,f^2\rangle)}$ and $U^+_{\mathcal{F}(\langle f^1,f^2\rangle)}$. Let $A = \{\langle u_n^1, u_n^2\rangle\}_{n\in\omega}$ be a monotonic directed set and $\bigvee^{\uparrow} A =< u^1, u^2 >$. We check that if u_n^1, u_n^2, u^1 and u^2 are continuous, then $\bigvee^{\uparrow} \mathcal{F}(\langle u_n^1, u_n^2\rangle) = \mathcal{F}(\langle u^1, u^2\rangle)$. By monotonicity of $\mathcal{F}$, $\mathcal{F}(\langle u^1, u^2\rangle) \sqsupseteq \mathcal{F}(\langle u_n^1, u_n^2\rangle)$ for all $n \in \omega$. Hence $\mathcal{F}(\langle u^1, u^2\rangle) \sqsupseteq \bigvee^{\uparrow} \mathcal{F}(\langle u_n^1, u_n^2\rangle)$. To prove the inclusion $\bigvee^{\uparrow} \mathcal{F}(\langle u_n^1, u_n^2\rangle) \sqsupseteq \mathcal{F}(\langle u^1, u^2\rangle)$, it is enough to check that for (x,t) such that $(x,t) \in U^-_{\mathcal{F}(\langle u^1,u^2\rangle)}$ there exists $n \in \omega$ with $(x,t) \in U^-_{\mathcal{F}(\langle u_n^1,u_n^2\rangle)}$. Suppose the contrary. For some (x,t), $(x,t) \in U^-_{\mathcal{F}(\langle u^1,u^2\rangle)}$, but for all $n \in \omega$ we have $(x,t) \notin U^-_{\mathcal{F}(\langle u_n^1,u_n^2\rangle)}$. Let us find $\epsilon > 0$ such that the condition $(x,t) \in U^-_{\mathcal{F}(\langle u^1,u^2\rangle)}$. For all n we have a sequence $\{h_m^n\}_{m\in\omega}$ which is closely approximating to $\langle u_n^1, u_n^2\rangle$ and $B((x,t),\epsilon) \not\subseteq U^-_{F(h_m^n)}$ for infinitely great m. From the set $\{h_m^n\}_{n\in\omega, m\in\omega}$ we can extract a sequence $\{\tau_n\}_{n\in\omega}$ which is closely approximating to $\langle u^1, u^2\rangle$ and $B((x,t),\epsilon) \not\subseteq U^-_{F(\tau^n)}$ for $n \in \omega$. This is a contradiction with the choice of ϵ.

Now we define F^* for $\langle f^1, f^2\rangle \in I_f^0([a,b])$ by the following rule: $F^*(\langle f^1, f^2\rangle) = \bigvee^{\uparrow} \mathcal{F}(\langle f_n^1, f_n^2\rangle)$, where $\bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle = \langle f^1, f^2\rangle$ and f_n^1, f_n^2 are continuous, $n \in \omega$. Let us prove correctness of this definition. Suppose $\bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle = \bigvee^{\uparrow} \langle u_n^1, u_n^2\rangle = \langle f^1, f^2\rangle$. For a fix n we have $\langle u^1, u^2\rangle \sqsubseteq \bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle$ and $\langle u_n^1, u_n^2\rangle = \bigvee^{\uparrow} g.l.b.(\langle f_k^1, f_k^2\rangle, \langle u_n^1, u_n^2\rangle)$. By the property of $\mathcal{F}$, $\mathcal{F}(\langle u_n^1, u_n^2\rangle) = \bigvee^{\uparrow} \mathcal{F}(g.l.b.(\langle f_k^1, f_k^2\rangle, \langle u_n^1, u_n^2\rangle))$. By monotonicity of $\mathcal{F}$, $\mathcal{F}(g.l.b.(\langle f_k^1, f_k^2\rangle, \langle u_n^1, u_n^2\rangle)) \sqsubseteq \mathcal{F}(\langle f_k^1, f_k^2\rangle)$. So $\mathcal{F}(\langle u_n^1, u_n^2\rangle) \sqsubseteq \bigvee^{\uparrow} \mathcal{F}(\langle f_k^1, f_k^2\rangle)$. As a consequence, $\bigvee^{\uparrow} \mathcal{F}(\langle u_n^1, u_n^2\rangle) \sqsubseteq \bigvee^{\uparrow} \mathcal{F}(\langle f_k^1, f_k^2\rangle)$.

Similarly we can check inclusion $\bigvee^{\uparrow} \mathcal{F}(\langle u_n^1, u_n^2\rangle) \sqsupseteq \bigvee^{\uparrow} \mathcal{F}(\langle f_k^1, f_k^2\rangle)$. Monotonicity of F^* follows from monotonicity of $\mathcal{F}$. Now we prove continuity of F^*. Let the sequence $\{\langle f_n^1, f_n^2\rangle\}_{n\in\omega}$ be monotonic, and $\bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle = \langle f^1, f^2\rangle$ for $\langle f^1, f^2\rangle \in I_f^0([a,b])$. By the property of bases, $\langle f_n^1, f_n^2\rangle = \bigvee^{\uparrow} \kappa_n^m$ where $\kappa_n^m \in I_{f,0}^0([a,b])$. Put $\underset{i\leq i\leq n}{l.u.b.}\{\kappa_n^m\} = \lambda_n^m$. Then $\langle f_n^1, f_n^2\rangle = \bigvee^{\uparrow} \lambda_n^m$ and $\lambda_n^{m+1} \geq \lambda_n^m$. We have $\bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle = \bigvee^{\uparrow}\bigvee^{\uparrow} \lambda_n^m$.

Let us check that $\bigvee^{\uparrow} F^*(\langle f_n^1, f_n^2\rangle) = F^*(\bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle)$. We have $F^*(\bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle) = F^*(\bigvee^{\uparrow}\bigvee^{\uparrow} \lambda_n^m) = \bigvee^{\uparrow}\bigvee^{\uparrow} F^*(\lambda_n^m) \geq \bigvee^{\uparrow} F^*(\lambda_n^m) = F^*(\bigvee^{\uparrow} \lambda_n^m) = F^*(\langle f_n^1, f_n^2\rangle)$. So $\bigvee^{\uparrow} F^*(\langle f_n^1, f_n^2\rangle) \sqsubseteq F^*(\bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle)$.

Moreover, $F^*(\bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle) = \bigvee^{\uparrow}\bigvee^{\uparrow} F*(\lambda_n^m)$ and $F*(\lambda_n^m) \sqsubseteq F*(\langle f_n^1, f_n^2\rangle)$. So $\bigvee^{\uparrow} F^*(\langle f_n^1, f_n^2\rangle) \sqsupseteq F^*(\bigvee^{\uparrow} \langle f_n^1, f_n^2\rangle)$. Continuity of F^* is proved, and so F^* is a required one.

□

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