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Model checking μ-calculus in well-structured transition systems

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Abstract. We study the model checking problem for fixpoint logics in well-structured multiaction transition systems. P.A. Abdulla et al. (1996) and Finkel & Schnoebelen (2001) examined the decidability problem for liveness (reachability) and progress (eventuality) properties in well-structured single action transition systems. Our main result is as follows: the model checking problem is decidable for disjunctive formulae of the propositional μ-Calculus of D. Kozen (1983) in well-structured transition systems where propositional variables are interpreted by upward cones. We also discuss the model checking problem for the intuitionistic modal logic of Fisher Servi (1984) extended by least fixpoint.

Keywords: model checking, μ-Calculus, intuitionistic modal logic.

1. Well-preordered transition systems

Let D be a set. An equivalence is a reflexive, transitive and symmetric binary relation on D. A partial order is a reflexive, transitive, and antisymmetric binary relation on D. A preorder (synonym: quasi-order) is a reflexive and transitive binary relation on D. A well-preorder (synonym: well-quasi-order) is a preorder $\preceq$ where every infinite sequence d_0, ... d_i, ... of elements of D contains a pair of elements d_m and d_n so that $m < n$ and $d_m \preceq d_n$.

Every partial order is a preorder and, vise versa, almost every finite preorder is a partial order [6]:

$$\lim_{n\to\infty} \frac{|\{(D,R)\ :\ |D| = n\ \&\ R \text{ is a partial order on } D\}|}{|\{(D,R)\ :\ |D| = n\ \&\ R \text{ is a preorder on } D\}|} = 1.$$

Every finite preorder is automatically a well-preorder. Every finite equivalence relation is also a well-preorder. It is also known that there exist $n^{O(1)} \times 2^{\frac{n^2}{4}+\frac{3n}{2}}$ partial orders on n elements [7, 6]. It implies the following proposition.

Proposition 1. *There exist $n^{O(1)} \times 2^{\frac{n^2}{4}+\frac{3n}{2}}$ well-preorders on n-element sets.*

Let $(D, \preceq)$ be a well-preordered set (i.e., a set D provided with a well-preorder $\preceq$). An ideal (synonym: cone) is an upward closed subset of D, i.e. a set $I \subseteq D$ such that for all $d', d'' \in D$, if $d' \preceq d''$ and $d' \in I$ then $d'' \in I$. Every $d \in D$ generates the upward cone $(\uparrow d) \equiv \{e \in D : d \preceq e\}$. For every set $S \subseteq D$ and every element $d \in S$, d is a minimal element of S iff for every element $s \in S$ either $d \preceq s$ or d and s are non-comparable. For every subset $S \subseteq D$, the set of its minimal elements is $min(S)$. For every subset $S \subseteq D$, a basis of S is a subset $B \subset S$ such that for every $s \in S$ there exists an element $b \in B$ such that $b \preceq s$.

Let us present some algebraic properties of well-preorders that are easy to prove [1, 4]. Let us fix for simplicity a well-preordered set $(D, \preceq)$. First, $(D, \preceq)$ is well-founded, i.e. infinite strictly decreasing sequences of elements of D are impossible; moreover, every infinite sequence in $(D, \preceq)$ contains an infinite non-decreasing subsequence. Next, every subset $S \subseteq D$ provided with the preorder $\preceq$ also forms another well-preordered set $(S, \preceq)$. Third, every $S \subseteq D$ has a finite basis that consists of the set of the minimal elements $min(S)$; in particular, every ideal I has a finite basis $min(I)$, and $I = \cup_{d \in min(I)}(\uparrow d)$. Finally, every non-decreasing sequence of ideals $I_0 \subseteq ... \subseteq I_i \subseteq ...$ eventually stabilizes, i.e., there is some $k \geq 0$ such that $I_m = I_n$ for all $m, n \geq k$.

Let *Act* be a fixed finite alphabet of action symbols. A transition system (synonym: Kripke frame) is a tuple (D, R), where the domain D is a non-empty set of elements that are called states, and the interpretation R is a total mapping $R : Act \to 2^{D \times D}$. A run (in the frame) is a maximal sequence of states $s_1 ... s_i s_{i+1} ...$ such that for all adjacent states within the sequence $(s_i, s_{i+1}) \in R(a)$ for some $a \in Act$.

A well-preordered transition system (WPTS) is a triple $(D, \preceq, R)$ such that $(D, \preceq)$ is a well-preordered set and (D, R) is a Kripke frame. We are most interested in well-preordered transition systems with decidable and compatible well-preorders and interpretations. The decidability condition for the well-preorder is straightforward: $\preceq \subseteq D \times D$ is decidable. The decidability condition for interpretations of action symbols and compatibility conditions for well-preorders and interpretations of action symbols are discussed below.

Let $(D, \preceq, R)$ be a WPTS and $a \in Act$ be an action symbol. We consider the following decidable condition for the interpretation $R(a)$ of the action symbol $a \in Act$: the function $\lambda\ s \in D\ .\ min\{t : t \xrightarrow{R(a)} s\}$ is computable. We refer to this condition as tractable past.

Again, let $(D, \preceq, R)$ be a WPTS and $a \in Act$ be an action symbol. There are 4 options for strong compatibility of the well-preorder $\preceq$ and the interpretation $R(a)$ of the action symbol $a \in Act$. They are represented in Tables 1 and 2. The terminology used in these tables is explained in the

Table 1. (Future) Fisher Servi conditions

notation	(future) upward	(future) downward
logic	$\forall s'_1, s''_1, s'_2\ \exists s''_2\ :$ $s'_1 \xrightarrow{R(a)} s''_1\ \&\ s'_1 \preceq s'_2\ \Rightarrow$ $\Rightarrow s'_2 \xrightarrow{R(a)} s''_2\ \&\ s''_1 \preceq s''_2$	$\forall s'_1, s'_2, s''_2\ \exists s''_1\ :$ $s'_2 \xrightarrow{R(a)} s''_2\ \&\ s'_1 \preceq s'_2 \Rightarrow$ $\Rightarrow s'_1 \xrightarrow{R(a)} s''_1\ \&\ s''_1 \preceq s''_2$
diagram	$\begin{array}{ccc} s''_1 & .\preceq. & s''_2 \\ \uparrow & & \vdots \\ s'_1 & \preceq & s'_2 \end{array}$	$\begin{array}{ccc} s''_1 & .\preceq. & s''_2 \\ \vdots & & \uparrow \\ s'_1 & \preceq & s'_2 \end{array}$
algebraic	$\preceq^{-} \circ R(a)\ \subseteq\ R(a) \circ \preceq^{-}$	$\preceq \circ R(a)\ \subseteq R(a) \circ \preceq$

following three paragraphs.

The adjectives "upward" and "downward" have been introduced by [4]; they have explicit mnemonics. The adjective "strong" has also been introduced by [4]; it refers to a single step of action $R(a)$ that interprets the corresponding action symbol a. In accordance with [4], one can define the transitive, reflexive and "plain" compatibility by using the transitive closure $\big(R(a)\big)^+$, the reflexive closure $\big(=\ \cup\ R(a)\big)$ and the reflexive-transitive closure $\big(R(a)\big)^*$ instead of the single step $R(a)$.

The Fisher Servi conditions are due to intuitionistic modal logic **FS** suggested by G. Fisher Servi [5] (see also [10] and [3]). Semantics of **FS** is defined in partially ordered transition systems $(D, \preceq, R)$, where $\preceq$ is a partial order which is upward and downward compatible with R.

Finally, adjectives "future" and "past" are used to distinguish between compatibility conditions in Tables 1 and 2. The former is optional, it is about states after an action, i.e., future states. The latter is obligatory, it is about states before an action, i.e., past states.

Let M be a WPTS. We say that M has tractable past, iff it enjoys this property for every action symbol $a \in Act$. Let us fix a particular compatibility property from Tables 1 and 2; we say that M has this property, iff it enjoys it for every action symbol $a \in Act$.

An upward compatible well-preordered transition system with tractable past and decidable preorder is said to be a well-structured transition system (WSTS).

Extensive case study and some generic examples of single action[1] WPTSes with upward compatible preorders and interpretations can be found in the foundational papers [1, 4]. Maybe, the most popular case study is vector-addition systems (i.e., Petri nets) with the component-wise inequality (the subset relation on markings, respectively), while the most interesting

[1] i.e., when $|Act| = 1$

Table 2. Past Fisher Servi conditions

notation	past upward	past downward
logic	$\forall s'_1, s''_1, s''_2\ \exists s'_2\ :$ $s'_1 \xrightarrow{R(a)} s''_1\ \&\ s''_1 \preceq s''_2 \Rightarrow$ $\Rightarrow s'_2 \xrightarrow{R(a)} s''_2\ \&\ s'_1 \preceq s'_2$	$\forall s''_1, s''_2, s'_2\ \exists s'_1\ :$ $s'_2 \xrightarrow{R(a)} s''_2\ \&\ s''_1 \preceq s''_2 \Rightarrow$ $\Rightarrow s'_1 \xrightarrow{R(a)} s''_1\ \&\ s'_1 \preceq s'_2$
diagram	$\begin{array}{ccc} s''_1 & \preceq & s''_2 \\ \uparrow & & \vdots \\ s'_1 & .\preceq. & s'_2 \end{array}$	$\begin{array}{ccc} s''_1 & \preceq & s''_2 \\ \vdots & & \uparrow \\ s'_1 & .\preceq. & s'_2 \end{array}$
algebraic	$\preceq^- \circ R(a)^- \ \subseteq\ R(a)^- \circ \preceq^-$	$\preceq \circ R(a)^- \ \subseteq\ R(a)^- \circ \preceq$

generic examples are

- finite transition systems with equality,
- transition systems with comparison of length of the longest runs.

Paper [4] has proved that all these single action WPTSes enjoy future upward strong *strict*[2] compatibility.

We revise the generic examples for multiaction transition systems[3] and Fisher Servi compatibility, i.e., for simultaneous upward and downward strong compatibility. First, every frame for the intuitionistic modal logic **FS** is a automatically Fisher Servi compatible WPTS with single action.

Next, there are close relations between compatibility and (bi)simulation [9, 12]. Let $(D, \preceq, R)$ be a WPTS. One can see that

- future upward compatibility states that the well-preorder $\preceq$ is a simulation relation on the states of the transition system (D, R);
- future downward compatibility states that the inverse $\preceq^-$ of the well-preorder $\preceq$ is a simulation relation on the states of the transition system (D, R).

These observations lead to the following proposition and corollary.

Proposition 2. *Every transition system (D, R) provided with any bisimulation $\simeq$ on the states in D forms a Fisher Servi compatible WPTS $(D, \simeq, R)$. In particular, (D, R) provided with equality forms a Fisher Servi compatible WPTS $(D, =, R)$.*

[2]i.e. admits $\prec$ ($\equiv \preceq\ \&\ \not\succeq$) as well as $\preceq$

[3]i.e., when $|Act| \geq 1$

Corollary 1. *Every finite transition system (D, R) provided with any bisimulation $\simeq$ on the states in D forms a downward compatible well-structured transition system $(D, \simeq, R)$ with tractable past. In particular, a finite (D, R) provided with equality forms a downward compatible well-structured transition system $(D, =, R)$.*

2. Propositional μ-calculus

The μ-Calculus of D.Kozen (μC) [8] is a very powerful propositional program logic with fixpoints. It is widely used for specification and verification of properties of finite state systems. (Please refer to [11] for the elementary introduction to μC. The comprehensive definition of μC can be found, for example, in a recent textbook [2].) We would like to point out that some authors denote the *mu*-Calculus with the single Action symbol by $L_{\Box\Diamond\mu\nu}$ since in the single action settings it becomes a propositional modal logic with two modalities ($\Box$ and $\Diamond$) extended by fixpoints (μ and ν). If we assume a standard duality between modalities $\Box$ and $\Diamond$ and between fixpoints μ and ν, then $L_{\Box\Diamond\mu\nu}$ becomes $\mu\mathbf{K}$ – the basic propositional modal logic $\mathbf{K}$ extended by fixpoints.

The syntax of μC consists of formulae. Let *Prp* be an alphabet of propositional variables which is disjoint with the alphabet of action symbols *Act* fixed above. A context-free definition of μC formulae is as follows:

$$\phi ::= p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid ([a]\phi) \mid (\langle a\rangle\phi) \mid (\nu\, p.\, \phi) \mid (\mu\, p.\, \phi),$$

where metavariables ϕ, p, and a range over formulae, propositional variables and action symbols. The only context constraint is the following: no instances of bound (by μ or ν) propositional variables are in the range of odd number of negations.

The semantics of μC is defined in labeled transition systems (synonym: Kripke models). A model is a triple (D, R, V), where (D, R) is a Kripke frame, and the valuation V is another total mapping $V : Prp \to 2^D$. In every model $M = (D, R, V)$, for every formula ϕ, the semantics $M(\phi)$ is a subset of the domain D that is defined by induction on the formula structure:

- $M(p) = V(p)$, $M(\neg\psi) = D \setminus M(\psi)$,
  $M(\psi' \wedge \psi'') = M(\psi') \cap M(\psi'')$, $M(\psi' \vee \psi'') = M(\psi') \cup M(\psi'')$,
- $M([a]\psi) = \{\ s\ :\ t \in M(\psi)$ for every t such that $(s, t) \in R(a)\ \}$,
  $M(\langle a\rangle\psi) = \{\ s\ :\ t \in M(\psi)$ for some t such that $(s, t) \in R(a)\ \}$,
- $M(\nu p.\psi) =$ the greatest fixpoint of the mapping
  $\lambda S \subseteq D\ .\ \Big(M_{S/p}(\psi)\Big)$,

$M(\mu p.\psi) =$ the least fixpoint of the mapping
$\lambda S \subseteq D \;.\; \Big(M_{S/p}(\psi)\Big)$,

where metavariables ψ, ψ', ψ'', p, and a range over formulae, propositional variables and action symbols, and $M_{S/p}$ denotes the model that agrees with M everywhere but p: $V_{S/p}(p) = S$.

A propositional variable is said to be a propositional constant in a formula iff it is free in the formula. A formula is said to be in the normal form iff negation is applied to propositional constants in the formula only. A formula is said to be positive iff it is negation-free. Due to the standard De Morgan laws

$$(\neg(\neg\phi)) \leftrightarrow \phi$$
$$(\neg(\phi \wedge \psi)) \leftrightarrow ((\neg\phi) \vee (\neg\psi)) \qquad (\neg(\phi \vee \psi)) \leftrightarrow ((\neg\phi) \wedge (\neg\psi))$$
$$(\neg(\langle a\rangle\phi)) \leftrightarrow ([a](\neg\phi)) \qquad (\neg([a]\phi)) \leftrightarrow (\langle a\rangle(\neg\phi))$$
$$(\neg(\mu p.\phi)) \leftrightarrow (\nu p.(\neg(\phi)_p^{(\neg p)}) \qquad (\neg(\nu p.\phi)) \leftrightarrow (\mu p.(\neg(\phi)_p^{(\neg p)})$$

every formula of μC is equivalent to some formula in the normal form that can be constructed in polynomial time. (Here and throughout the paper X_Z^Y stays for substitution of Y instead of all instances of Z into X.)

We are especially interested in the fragment of the μ-Calculus that comprises the disjunctive formulae, i.e., formulae without negations $\neg$, conjunctions $\wedge$, and "infinite conjunctions" $[\,]$ and ν. A context-free definition of these formulae is the following:

$$\phi \ ::= \ p \ | \ (\phi \vee \phi) \ | \ (\langle a\rangle\phi) \ | \ (\mu\, p.\, \phi),$$

where metavariables ϕ, p, and a range over formulae, propositional variables and action symbols. We can remark that liveness and progress properties are easy to present in this fragment: $\mathbf{EF}p \leftrightarrow \mu q.(p \vee \langle next\rangle q)$ and $\mathbf{AF}p \leftrightarrow \mu q.(p \vee [next]q)$, where *next* is the single implicit action symbol of CTL.

Another logic that we use in our studies is the Fisher Servi intuitionistic modal logic **FS** [5, 10, 3]. The syntax of **FS** consists of formulae that are constructed from propositional variables Prp in accordance with the following context-free definition:

$$\phi \ ::= \ p \ | \ (\neg\phi) \ | \ (\phi \to \phi) \ | \ (\phi \wedge \phi) \ | \ (\phi \vee \phi) \ | \ (\Box\phi) \ | \ (\Diamond\phi),$$

where metavariables ϕ and p range over formulae and propositional variables. The semantics of **FS** is defined in intuitionistic Kripke models. A model of this kind is a quadruple $(D, \preceq, R, V)$, where the domain D is a nonempty set of states, $\preceq$ is a partial order on D, the interpretation R interprets the single implicit action symbol (say *next*) by a binary relation $R(next) \subseteq D \times D$ in an upward and downward compatible manner with $\preceq$, and the valuation V is a total mapping $V : Prp \to \{I \subseteq D : I$ is a cone in $(D, \preceq)\}$.

In every model $M = (D, \preceq, R, V)$, for every formula ϕ, the semantics $M(\phi)$ is a subset of the domain D that is defined by induction on the formula structure:

- $M(p) = V(p)$, $M(\neg\psi) = \{s \ : \ (\uparrow s) \cap M(\psi) = \emptyset\}$,
  $M(\psi' \to \psi'') = \{s \ : \ (\uparrow s) \cap M(\psi') \subseteq M(\psi'')\}$,
  $M(\psi' \wedge \psi'') = M(\psi') \cap M(\psi'')$, $M(\psi' \vee \psi'') = M(\psi') \cup M(\psi'')$,
- $M(\Box\psi) = \{\ s \ : \ (\uparrow t) \subseteq M(\psi)$ for every t such that $(s, t) \in R(next)\ \}$,
  $M(\Diamond\psi) = \{\ s \ : \ t \in M(\psi)$ for some t such that $(s, t) \in R(next)\ \}$,

where metavariables ψ, ψ', ψ'', and p range over formulae and propositional variables, respectively. (Sic! In contrast to classical modal logics, there is no standard duality between $\Box$ and $\Diamond$ in intuitionistic modal logic.)

Please refer to papers [5, 10, 3] for finite model property, axiomatization, and decidability issues of **FS**, but let us define a variant μ**FS** of **FS** with multiactions and fixpoints as follows. The syntax of μ**FS** coincides with the syntax of μC. The semantics of μ**FS** is defined in models that are partially ordered Fisher Servi compatible labeled transition systems. A model of this kind is a quadruple $(D, \preceq, R, V)$, where the domain D is a nonempty set of states, $\preceq$ is a partial order on D, the interpretation R is a total mapping $R : Act \to 2^{D\times D}$ that interprets every action symbol $a \in Act$ by a binary relation $R(a) \subseteq D \times D$ in an upward and downward compatible manner with $\preceq$, and the valuation V is a total mapping $V : Prp \to \{I \subseteq D \ : \ I$ is a cone in $(D, \preceq)\}$ (i.e., it interprets every propositional variable $p \in Prp$ by some ideal in $(D, \preceq)$).

In every model $M = (D, \preceq, R, V)$, for every formula ϕ, the semantics $M^{Int}(\phi)$ is a subset of the domain D that is defined by induction on the formula structure:

- $M^{Int}(p) = V(p)$, $M^{Int}(\neg\psi) = \{s \ : \ (\uparrow s) \cap M^{Int}(\psi) = \emptyset\}$,
  $M^{Int}(\psi' \to \psi'') = \{s \ : \ (\uparrow s) \cap M^{Int}(\psi') \subseteq M^{Int}(\psi'')\}$,
  $M^{Int}(\psi' \wedge \psi'') = M^{Int}(\psi') \cap M^{Int}(\psi'')$,
  $M^{Int}(\psi' \vee \psi'') = M^{Int}(\psi') \cup M^{Int}(\psi'')$,
- $M^{Int}([a]\psi) = \{\ s \ : \ (\uparrow t) \subseteq M^{Int}(\psi)$ for every t such that $(s, t) \in R(a)\ \}$,
  $M^{Int}(\langle a\rangle\psi) = \{\ s \ : \ t \in M^{Int}(\psi)$ for some t such that $(s, t) \in R(a)\ \}$,
- $M^{Int}(\nu p.\psi) =$ the greatest fixpoint of the mapping $\lambda S \subseteq D \ . \ \left(M^{Int}_{S/p}(\psi)\right)$,
  $M^{Int}(\mu p.\psi) =$ the least fixpoint of the mapping $\lambda S \subseteq D \ . \ \left(M^{Int}_{S/p}(\psi)\right)$,

where metavariables ψ, ψ', ψ'', p, and a range over formulae, propositional variables and action symbols, and $M^{Int}_{S/p}$ denotes the model that agrees with M^{Int} everywhere but p: $V_{S/p}(p) = S$.

The following proposition is standard for intuitionistic logic.

Proposition 3. *For every* μ**FS** *model* M*, for every* μ**FS** *formula* ϕ*, the intuitionistic semantics* $M^{Int}(\phi)$ *is an upward cone.*

We are especially interested in the fragment of μ**FS** that comprises the disjunctive formulae, i.e., formulae without negations $\neg$, implications $\rightarrow$, conjunctions $\wedge$, and "infinite conjunctions" [ ] and ν, i.e., they coincide with the disjunctive formulae of μC. It is easy to observe that clauses responsible for semantics of the disjunctive formulae in μC and in μ**FS** also coincide. It leads to the following proposition.

Proposition 4. *For every* μ**FS** *model* M*, for every disjunctive* μ**FS** *formula* ϕ*, the intuitionistic semantics* $M^{Int}(\phi)$ *coincides with the classical semantics* $M(\phi)$*.*

3. The main result and conclusion

A well-structured labeled transition system is a quadruple $(D, \preceq, R, V)$, where (D, R, V) is a labeled transition system, and $(D, \preceq, R)$ is a well-structured transition system. An ideal-based model is a well-structured labeled transition system $(D, \preceq, R, V)$, where $V : Prp \rightarrow \{I \subseteq D : I$ is a cone in $(D, \preceq)\}$, i.e., it interprets every propositional variable $p \in Prp$ by some ideal in $(D, \preceq)$. In particular, every μ**FS** model is an ideal-based model that is also downward compatible.

Proposition 5. *For every positive formula* ϕ *of the* μ*-Calculus without conjunctions* $\wedge$*, boxes* [ ]*, and greatest fixpoints* ν*, for every ideal-based model* M*, the semantics* $M(\phi)$ *is an ideal. Moreover, if valuations of all propositional constants in* ϕ *are defined by their finite bases, then some finite basis for* $M(\phi)$ *is computable.*

Proof. For every model M, the semantics for positive formulae without infinite conjunctions in M is defined in accordance with the following clauses:

- $M(p) = V(p)$, $M(\psi' \vee \psi'') = M(\psi') \cup M(\psi'')$,
- $M(\langle a \rangle \psi) = \{\ s\ :\ t \in M(\psi)$ for some t such that $(s, t) \in R(a)\ \}$,
- $M(\mu p.\psi) =$ the least fixpoint of the mapping $\lambda S \subseteq D\ .\ \left(M_{S/p}(\psi)\right)$,

where metavariables ψ, ψ', ψ'', p, and a range over formulae, propositional variables and action symbols. We prove both claims of the proposition by induction on the structure of a positive disjunctive formula ϕ on the base of the semantics defined above. Induction basis is quite trivial since it is about propositional constants.

The first inductive case is $\phi \equiv (\psi' \vee \psi'')$. For every ideal-based model $M = (D, \preceq, R, V)$, if $M(\psi')$ and $M(\psi'')$ are ideals, then $M(\phi)$ is an ideal too, since a union of ideals is also an ideal; if B' and B'' are finite bases of ideals $M(\psi')$ and $M(\psi'')$, then $B' \cup B''$ is a computable finite basis for $M(\psi)$.

The next inductive case is $\phi \equiv (\langle a \rangle \psi)$. Let $M = (D, \preceq, R, V)$ be an ideal-based model and let $M(\psi)$ be an ideal. Semantics of ϕ in M is $\{s : t \in M(\psi)$ for some t such that $(s,t) \in R(a)\}$. Let s be a state in $M(\phi)$ and t be a corresponding state in $M(\psi)$ such that $(s,t) \in R(a)$. Let s' be any state such that $s' \succeq s$. In accordance with upward compatibility, there exists a state t' such that $(s',t') \in R(a)$ and $t' \succeq t$. This state t' is in $M(\psi)$, since $M(\psi)$ is an ideal that contains t. It implies that s is in $M(\phi)$. Thus $M(\phi)$ is an ideal too. If $\{s_0, ...s_n\}$ is a finite basis of $M(\psi)$, then (in accordance with tractable past) $min\{t : (t, s_i) \in R(a)\}$ is computable for every $i \in [0..n]$. Hence a finite set $\bigcup_{i=0}^{i=n} min\{t : (t, s_i) \in R(a)\}$ is a computable finite basis for $M(\phi)$.

The last inductive case is $\phi \equiv (\mu p.\psi)$. Let $M = (D, \preceq, R, V)$ be an ideal-based model. In accordance with the Tarski–Knaster theorem [2, 11], $M(\phi) = V^\alpha(p)$, where α is the ordinal number of D, $V^0 = V_{\emptyset/p}$ and

$$M^0 = (D, \preceq, R, V^0), V^{(\beta+1)} = V_{M^\beta(\psi)/p} \text{ and } M^{\beta+1} = (D, \preceq, R, V^{\beta+1})$$

for every ordinal $\beta < \alpha$, $V^{(\beta)} = \bigcup_{\gamma<\beta} V^\gamma$ and $M^\beta = (D, \preceq, R, V^\beta)$ for every limit ordinal $\beta < \alpha$. One can observe that the propositional variable p is a propositional constant in ψ. In accordance with induction hypothesis for ψ, $M^\beta(\psi)$ is some ideal I_β for every ordinal $\beta < \alpha$. The sequence of this ideals is non-decreasing in accordance with the Tarski–Knaster theorem. Hence it stabilizes on some finite ordinal $n < \alpha$ in well-preordered transition system M. In other words, $M(\phi) = I_n$ as soon as $I_n = I_{n+1}$, i.e., $M(\phi)$ is an ideal. Some finite bases are computable for all I_i, $i \in [0..n]$, and, in particular, some finite basis is computable for $M(\phi)$. □

Let $\mathcal{M}$ be a class of models, Φ be a class of formulae. The model checking problem for $\mathcal{M}$ and Φ is to decide the following set $\big\{ (\phi, M, s) : \phi \in \Phi,\ M \in \mathcal{M}$ and $s \in M(\phi)\big\}$. The following theorem is a corollary from Propositions 4 and 5.

Theorem 1. *The model checking problem is decidable for the ideal-based models and the disjunctive formulae of the propositional μ-Calculus. It is* *also decidable for the disjunctive formulae of the intuitionistic modal logic with least fixpoints μ**FS** in the models with tractable past.*

Let us conclude with some topics for further research.

A so-called Ubiquity Theorem 11 from [4] has stated that every single action transition system (D, R) can be provided with some well-preorder $\leq_T$ so that $(D, \leq_T, R)$ becomes a strict strong compatible WPTS. It follows from Proposition 2 that it is possible to adopt the equality $=$ as $\leq_T$. But the original proof of the Ubiquity Theorem in [4] has suggested another well-preorder on elements of D: $s' \leq_T s''$ iff the length of the longest run starting in s' is not greater than the length of the longest run starting in s''. The problem is: how to generalize the well-preorder $\leq_T$ for the multiaction case in an upward compatible manner, and in the simultaneous upward and downward compatible manner?

The number of upward compatible WPTS's and Fisher Servi compatible WPTS's is another research topic closely related to ubiquity. Proposition 1 states that there exist $n^{O(1)} \times 2^{\frac{2}{4}+\frac{3n}{2}}$ finite well-preorders on n-element sets. The problem is: how many upward and/or downward compatible well-preordered transition systems exist? The problem is interesting for intuitionistic model logics, since frames for **FS** logic are precisely single action upward and downward compatible WPTS.

It is also interesting to extend the decidability Theorem 1 from disjunctive formulae to positive formulae in the classical (μC) and in the intuitionistic (μ**FS**) cases. In both cases, the semantics of formulae are cones. It is easy to see that if an ideal-based model is an upper semilattice with efficient procedure for the least upper bounds, then semantics of the conjunctions becomes computable in this model. The problem is: what conditions can guarantee computability of semantics in the ideal-based models for infinite conjunctions [ ] and ν?

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