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The derivation of the Saint–Venant equations

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Abstract. The water motion in real river-beds is described by the mathematical models that under appropriate assumptions are various approximations of hydrodynamic equations, or the Navier–Stokes equations. The mathematical model of a water flow, based on the laws of conservation of momentum and mass of fluid, was proposed by Saint–Venant. The Saint–Venant equations (the shallow-water equations) are often used in theoretical and applied studies of the unsteady water motion in free channels. In this paper, the rigorous deduction of the shallow-water equations from the Navier–Stokes equations using the recommendations and methods proposed in [1] is presented.

The following hypotheses and assumptions are used for the derivation of the Saint–Venant equations [2]:

1. The length of the watercourse is much greater than its depth and width;
2. The river-bed is a straight line, i.e. the centrifugal forces are absent;
3. The pressure inside the water flow obeys the hydrostatic law: $p = \rho g(\xi - z) + p_a$, where ξ is the excess of water level over its equilibrium state and p_a is the atmospheric pressure;
4. The cross-section of the water surface is horizontal;
5. The flow is smoothly varying, subcritical, i.e. it has a Froude number less than one, $\mathrm{Fr} = v^2/(gL) < 1$, where v is the characteristic velocity scale, g is the acceleration describing the action of external forces, and L is the typical size of the field where the flow is considered (the length or diameter of a tube);
6. The bottom slope $I(x)$ is small, so that $I = I(x) = \operatorname{tg} x$;
7. The water discharge $Q(x,t)$ and the free surface level $Z(x,t)$ are averaged over the width and the depth of the flow, where $Q = VF$, $V(x,t)$ is the flow velocity and F is its sectional area.

The Navier–Stokes equations are as follows:

$$\begin{aligned}
\frac{du}{dt} &= X - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 u, \qquad X = g\sin\alpha,\\
\frac{dv}{dt} &= Y - \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla^2 v, \qquad Y = 0,\\
\frac{dw}{dt} &= Z - \frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w, \qquad Z = -g\cos\alpha = -g_*.
\end{aligned} \tag{1}$$

Here (X, Y, Z) represent the acceleration vector of external forces, α is the angle of the slope of the axis x to the horizon, g is the acceleration of gravity.

Using the continuity equation for the incompressible fluid flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

and bearing in mind that $u = u(x(t), y(t), z(t))$, $v = v(x(t), y(t), z(t))$, and $w = w(x(t), y(t), z(t))$, we obtain

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\\
&= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\\
&= \frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z},\\
\frac{dv}{dt} &= \frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial (vv)}{\partial y} + \frac{\partial (vw)}{\partial z},\\
\frac{dw}{dt} &= \frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} + \frac{\partial (wv)}{\partial y} + \frac{\partial (ww)}{\partial z}.
\end{aligned}$$

Taking into account these formulas, the Navier–Stokes equations may be presented in the form

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} &= X - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 u,\\
\frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial (vv)}{\partial y} + \frac{\partial (vw)}{\partial z} &= Y - \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla^2 v,\\
\frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} + \frac{\partial (wv)}{\partial y} + \frac{\partial (ww)}{\partial z} &= Z - \frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w.
\end{aligned} \tag{2}$$

To obtain a differential equation of the unsteady turbulent water motion, we perform averaging with respect to time of the terms of system of equations (2). Thus,

$$\frac{1}{T_0}\int_0^{T_0}\frac{\partial u}{\partial t}dt+\frac{1}{T_0}\int_0^{T_0}\Big(\frac{\partial(uu)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}\Big)dt$$
$$=\frac{1}{T_0}\Bigg[\int_0^{T_0}X\,dt-\int_0^{T_0}\frac{1}{\rho}\frac{\partial P}{\partial x}\,dt+\int_0^{T_0}\nu\nabla^2u\,dt\Bigg].$$

Since $\rho=\text{const}$ and $\nu=\text{const}$ for an incompressible fluid, then

$$\frac{1}{T_0}\int_0^{T_0}X\,dt=\overline{X},\qquad \frac{1}{T_0}\int_0^{T_0}\frac{1}{\rho}\frac{\partial P}{\partial x}\,dt=\frac{1}{\rho}\frac{\partial\overline{P}}{\partial x},$$
$$\frac{1}{T_0}\int_0^{T_0}\frac{\partial u}{\partial t}\,dt=\frac{\partial\overline{u}}{\partial t},\qquad \frac{1}{T_0}\int_0^{T_0}\nu\nabla^2u\,dt=\nu\nabla^2\overline{u},$$

where $\overline{X}$ is the averaged value, $[0,T_0]$ is averaging segment, ∇^2 is Laplace operator. If we suppose that u',v',w' are oscillatory components of the velocity vector, then $u'=u-\overline{u}$, $v'=v-\overline{v}$, $w'=w-\overline{w}$, $uu=(\overline{u}+u')^2=\overline{u}\overline{u}+2\overline{u}u'+u'u'$, ...

Let us recall the averaging rules

- $\overline{f+g}=\overline{f}+\overline{g}$,
- $\overline{\overline{f}g}=\overline{f}\overline{g}$,
- $\overline{\overline{f}}=\overline{f}$, $\overline{f'}=\overline{f-\overline{f}}=0$,
- $\overline{\overline{f}\overline{g}}=\overline{f}\overline{g}$, $\overline{\overline{f}\cdot g'}=\overline{f}\,\overline{g'}=0$.

Thus, $\overline{uu}=\overline{u}\,\overline{u}+\overline{u'u'}$, $\overline{uv}=\overline{u}\,\overline{v}+\overline{u'v'}$, $\overline{uw}=\overline{u}\,\overline{w}+\overline{u'w'}$. Taking into account these expressions,

$$\frac{1}{T_0}\int_0^{T_0}\Big(\frac{\partial(uu)}{\partial x}+\frac{\partial(uv)}{\partial y}+\frac{\partial(uw)}{\partial z}\Big)dt$$
$$=\frac{\partial(\overline{u}^2)}{\partial x}+\frac{\partial(\overline{u'u'})}{\partial x}+\frac{\partial(\overline{u}\,\overline{v})}{\partial y}+\frac{\partial(\overline{u'v'})}{\partial y}+\frac{\partial(\overline{u}\,\overline{w})}{\partial z}+\frac{\partial(\overline{u'w'})}{\partial z}.$$

Carrying out analogous averaging for other equations, we transform the Navier–Stokes equations to the following form:

$$\begin{aligned}
\frac{\partial\overline{u}}{\partial t}+\frac{\partial\overline{u}^2}{\partial x}+\frac{\partial(\overline{u}\,\overline{v})}{\partial y}+\frac{\partial(\overline{u}\,\overline{w})}{\partial z}&=\overline{X}-\frac{1}{\rho}\frac{\partial\overline{P}}{\partial x}-R_x,\\
\frac{\partial\overline{v}}{\partial t}+\frac{\partial(\overline{v}\,\overline{u})}{\partial x}+\frac{\partial\overline{v}^2}{\partial y}+\frac{\partial(\overline{v}\,\overline{w})}{\partial z}&=\overline{Y}-\frac{1}{\rho}\frac{\partial\overline{P}}{\partial y}-R_y,\\
\frac{\partial\overline{w}}{\partial t}+\frac{\partial(\overline{w}\,\overline{u})}{\partial x}+\frac{\partial(\overline{w}\,\overline{v})}{\partial y}+\frac{\partial\overline{w}^2}{\partial z}&=\overline{Z}-\frac{1}{\rho}\frac{\partial\overline{P}}{\partial z}-R_z,
\end{aligned}\tag{3}$$

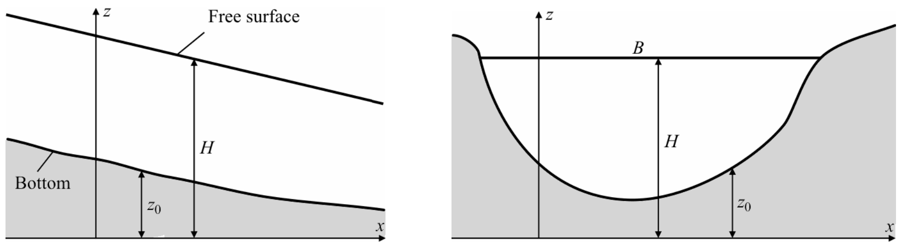
where

$$R_x = \frac{\partial(\overline{u'u'})}{\partial x} + \frac{\partial(\overline{u'v'})}{\partial y} + \frac{\partial(\overline{u'w'})}{\partial z} - \nu\nabla^2\overline{u},$$

$$R_y = \frac{\partial(\overline{u'v'})}{\partial x} + \frac{\partial(\overline{v'v'})}{\partial y} + \frac{\partial(\overline{v'w'})}{\partial z} - \nu\nabla^2\overline{v},$$

$$R_z = \frac{\partial(\overline{u'w'})}{\partial x} + \frac{\partial(\overline{v'w'})}{\partial y} + \frac{\partial(\overline{w'w'})}{\partial z} - \nu\nabla^2\overline{w}.$$

The arrangement of the axes x, y, z in the flow is shown in the figure.



Let us integrate the third equation of system (3) over z. With allowance for $\overline{X} = g\sin\alpha$, $\overline{Y} = 0$, and $\overline{Z} = -g_*$, we obtain

$$\frac{\overline{P}}{\rho} = \frac{\overline{P}_H}{\rho} + g_*(H - z) + \int_z^H R_z\,d\xi + \int_z^H \frac{\partial\overline{w}}{\partial t}\,d\xi + \int_z^H \frac{\partial(\overline{w}\,\overline{u})}{\partial x}\,d\xi + \int_z^H \frac{\partial(\overline{w}\,\overline{v})}{\partial y}\,d\xi + \overline{w}_H^2 - \overline{w}^2. \quad (4)$$

Here the subscript H denotes a value at $z = H$, i.e. at the free flow surface. In equation (4), we assume that the intersection of the free surface and the plane normal to the axis x is a straight line. It is evident that

$$\overline{w}_H = \overline{u}_H\frac{\partial H}{\partial x} + \frac{\partial H}{\partial t}. \quad (5)$$

Here the first summand appears due to the fact that x is not parallel to the free surface and the second one is due to the free surface movement.

Using the formula of differentiation under the integral

$$\int_\alpha^\beta \frac{\partial}{\partial\mu}\Phi(\lambda,\mu)\,d\lambda = \frac{d}{d\mu}\int_\alpha^\beta \Phi(\lambda,\mu)\,d\lambda - \Phi(\beta,\mu)\frac{d\beta}{d\mu} + \Phi(\alpha,\mu)\frac{d\alpha}{d\mu} \quad (6)$$

with $\alpha = z$ and $\beta = H(x,t)$, we transform some summands in (4) to the following form:

$$\int_z^H \frac{\partial \overline{w}}{\partial t}\, d\xi = \frac{\partial}{\partial t}\int_z^H \overline{w}\, d\xi - \overline{w}_H \frac{\partial H}{\partial t},$$

$$\int_z^H \frac{\partial (\overline{w}\,\overline{u})}{\partial x}\, d\xi = \frac{\partial}{\partial x}\int_z^H \overline{w}\,\overline{u}\, d\xi - \overline{w}_H \overline{u}_H \frac{\partial H}{\partial x},$$

$$\int_z^H \frac{\partial (\overline{w}\,\overline{v})}{\partial y}\, d\xi = \frac{\partial}{\partial y}\int_z^H \overline{w}\,\overline{v}\, d\xi.$$

Using (5) and the previous calculations, we rewrite (4) as follows:

$$\frac{\overline{P}}{\rho} = \frac{\overline{P}_H}{\rho} + g_*(H - z) + \int_z^H R_z\, d\xi + \frac{\partial}{\partial t}\int_z^H \overline{w}\, d\xi + \frac{\partial}{\partial x}\int_z^H \overline{w}\,\overline{u}\, d\xi + \frac{\partial}{\partial y}\int_z^H \overline{w}\,\overline{v}\, d\xi - \overline{w}^2. \tag{7}$$

It is supposed that the surface tension forces affect the stability of the flow, therefore we take them into account. Then $\overline{P}_H$ should be treated as a difference between the pressure under the surface of a thin film and the atmospheric pressure, that is

$$\overline{P}_H = -\frac{C}{L}, \qquad L = \left[1 + \left(\frac{\partial H}{\partial x}\right)^2\right]^{3/2} \Bigg/ \frac{\partial^2 H}{\partial x^2},$$

where L is the curvature radius of the free surface and C is the capillary constant.

Let us substitute $\overline{P}$ from (4) into the first equation of system (3). The resulting equation is averaged over the cross-section area of the flow F. In the future, the value of the cross-section area of the flow will be denoted by F instead of $|F|$. We will restrict our consideration to the case of the river-bed that is symmetric with respect to the plane xOz. Hence, all the integrals over F containing y-derivative are equal to zero. Thus, we have

$$\frac{1}{F}\int_F \left(\frac{\partial \overline{u}}{\partial t} + \frac{\partial \overline{u}^2}{\partial x} + \frac{\partial (\overline{u}\,\overline{v})}{\partial y} + \frac{\partial (\overline{u}\,\overline{w})}{\partial z} + R_x - \overline{X}\right) dF$$
$$= -\frac{1}{F}\int_F \left[\frac{\partial}{\partial x}\left(\frac{\overline{P}_H}{\rho} + g_*(H - z) + \int_z^H R_z\, d\xi + \frac{\partial}{\partial t}\int_z^H \overline{w}\, d\xi + \frac{\partial}{\partial x}\int_z^H \overline{w}\,\overline{u}\, d\xi + \frac{\partial}{\partial y}\int_z^H \overline{w}\,\overline{v}\, d\xi - \overline{w}^2\right)\right] dF,$$

or

$$\frac{1}{F}\int\limits_F\Bigg(\frac{\partial\overline{u}}{\partial t}+\frac{\partial\overline{u}^2}{\partial x}+\frac{\partial(\overline{u}\,\overline{v})}{\partial y}+\frac{\partial(\overline{u}\,\overline{w})}{\partial z}+R_x+\frac{\partial}{\partial x}\int\limits_z^H R_z\,d\xi+$$
$$\frac{\partial^2}{\partial x\partial t}\int\limits_z^H\overline{w}\,d\xi+\frac{\partial^2}{\partial x^2}\int\limits_z^H\overline{w}\,\overline{u}\,d\xi+\frac{\partial^2}{\partial x\partial y}\int\limits_z^H\overline{w}\,\overline{v}\,d\xi-\frac{\partial\overline{w}^2}{\partial x}\Bigg)dF$$
$$=\frac{1}{F}\int\limits_F\Big(\overline{X}-\frac{1}{\rho}\frac{\partial\overline{P}_H}{\partial x}-g_*\frac{\partial}{\partial x}(H-z)\Big)dF$$
$$=\frac{g_*}{F}\int\limits_F\Big(\frac{g\sin\alpha}{g_*}+\frac{C}{\rho g_*}\frac{\partial L^{-1}}{\partial x}-\frac{\partial H}{\partial x}+\frac{\partial z}{\partial x}\Big)dF$$
$$=g_*\Big(i+\sigma\frac{\partial L^{-1}}{\partial x}-\frac{\partial H}{\partial x}\Big).$$

For the sake of simplicity we introduced $i=\operatorname{tg}\alpha$ and $\sigma=\dfrac{C}{\rho g_*}$. Thus,

$$i+\sigma\frac{\partial L^{-1}}{\partial x}-\frac{\partial H}{\partial x}=\frac{1}{g_*F}\int\limits_F\Bigg(\frac{\partial\overline{u}}{\partial t}+\frac{\partial\overline{u}^2}{\partial x}+\frac{\partial(\overline{u}\,\overline{w})}{\partial z}+R_x+\frac{\partial}{\partial x}\int\limits_z^H R_z\,d\xi+$$
$$\frac{\partial^2}{\partial x\partial t}\int\limits_z^H\overline{w}\,d\xi+\frac{\partial^2}{\partial x^2}\int\limits_z^H\overline{w}\,\overline{u}\,d\xi-\frac{\partial\overline{w}^2}{\partial x}\Bigg)dF. \qquad (8)$$

Equation (8) should be transformed so that differentiation operation be placed outside the integral sign. To this end, we need to derive some formulas that are based on equation (6).

Bearing in mind that $z_0=z_0(x,y)$ is the equation of the wet river-bed surface and $H=H(x,t)$, $B=B(x,H)$ is the width of the free surface, we have

$$\int\limits_F\frac{\partial}{\partial x}\phi(x,y,z,t)\,dF=\int\limits_{-B/2}^{B/2}dy\int\limits_{z_0}^H\frac{\partial}{\partial x}\phi(x,y,z,t)\,dz.$$

Using formula (6), we obtain

$$\int\limits_{z_0}^H\frac{\partial}{\partial x}\phi(x,y,z,t)\,dz=\frac{\partial}{\partial x}\int\limits_{z_0}^H\phi(x,y,z,t)\,dz-$$
$$\phi(x,y,H,t)\frac{\partial H}{\partial x}+\phi(x,y,z_0,t)\frac{\partial z_0}{\partial x}.$$

Taking into consideration the fact that $z_0(x, B/2) = z_0(x, -B/2) = H$ and (6), the following can be derived:

$$\int\limits_F \frac{\partial}{\partial x}\phi(x,y,z,t)\,dF = \frac{\partial}{\partial x}\int\limits_F \phi(x,y,z,t)\,dF - \frac{\partial H}{\partial x}\int\limits_{-B/2}^{B/2}\phi(x,y,H,t)\,dy + \int\limits_{-B/2}^{B/2}\phi(x,y,z_0,t)\frac{\partial z_0}{\partial x}\,dy.$$

Replacing x-differentiation by t-differentiation yields

$$\int\limits_F \frac{\partial}{\partial t}\phi(x,y,z,t)\,dF = \frac{\partial}{\partial t}\int\limits_F \phi(x,y,z,t)\,dF - \frac{\partial H}{\partial t}\int\limits_{-B/2}^{B/2}\phi(x,y,H,t)\,dy.$$

If ϕ is an even function by y, that is $\phi(x,y,z,t) = \phi(x,-y,z,t)$, it is evident that

$$\int\limits_F \frac{\partial^2}{\partial x^2}\phi(x,y,z,t)\,dF = \frac{\partial^2}{\partial x^2}\int\limits_F \phi(x,y,z,t)\,dF - 2\frac{\partial H}{\partial x}\left[\frac{\partial}{\partial x}\int\limits_{-B/2}^{B/2}\phi(x,y,H,t)\,dy - 2\phi(x,B/2,H,t)\frac{\partial B}{\partial x}\right] - \frac{\partial^2 H}{\partial x^2}\int\limits_{-B/2}^{B/2}\phi(x,y,H,t)\,dy + 2\int\limits_{-B/2}^{B/2}\frac{\partial}{\partial x}\phi(x,y,z_0,t)\frac{\partial z_0}{\partial x}\,dy + \int\limits_{-B/2}^{B/2}\phi(x,y,z_0,t)\frac{\partial^2 z_0}{\partial x^2}\,dy;$$

$$\int\limits_F \frac{\partial^2}{\partial x\partial t}\phi(x,y,z,t)\,dF = \frac{\partial^2}{\partial x\partial t}\int\limits_F \phi(x,y,z,t)\,dF - \frac{\partial H}{\partial x}\left[\frac{\partial}{\partial t}\int\limits_{-B/2}^{B/2}\phi(x,y,H,t)\,dy - 2\phi(x,B/2,H,t)\frac{\partial B}{\partial t}\right] - \frac{\partial H}{\partial t}\left[\frac{\partial}{\partial x}\int\limits_{-B/2}^{B/2}\phi(x,y,H,t)\,dy - 2\phi(x,B/2,H,t)\frac{\partial B}{\partial x}\right] + \int\limits_{-B/2}^{B/2}\frac{\partial}{\partial t}\phi(x,y,z_0,t)\frac{\partial z_0}{\partial x}\,dy.$$

Using the previous calculations, we can rewrite the integrals from (8):

$$\int\limits_F \frac{\partial \overline{u}}{\partial t}\, dF = \frac{\partial}{\partial t}\int\limits_F \overline{u}\, dF - \frac{\partial H}{\partial t}\int\limits_{-B/2}^{B/2} \overline{u}_H\, dy = \frac{\partial}{\partial t}\int\limits_F \overline{u}\, dF - \overline{u}_H B \frac{\partial H}{\partial t},$$

$$\int\limits_F \frac{\partial \overline{u}^2}{\partial x}\, dF = \frac{\partial}{\partial x}\int\limits_F \overline{u}^2\, dF - \frac{\partial H}{\partial x}\int\limits_{-B/2}^{B/2} \overline{u}_H^2\, dy + \int\limits_{-B/2}^{B/2} \overline{u}_{z_0}^2 \frac{\partial z_0}{\partial x}\, dy,$$

$$\int\limits_F \frac{\partial (\overline{u}\,\overline{w})}{\partial z}\, dF = \int\limits_{-B/2}^{B/2} dy \int\limits_{z_0}^{H} \frac{\partial (\overline{u}\,\overline{w})}{\partial z}\, dz = \int\limits_{-B/2}^{B/2} \left[(\overline{u}\,\overline{w})_H - (\overline{u}\,\overline{w})_{z_0} \right] dy,$$

$$\int\limits_F \frac{\partial^2}{\partial x \partial t}\int\limits_{z_0}^{H} \overline{w}\, d\xi\, dF = \frac{\partial^2}{\partial x \partial t}\int\limits_F \int\limits_{z_0}^{H} \overline{w}\, d\xi\, dF -$$

$$\frac{\partial H}{\partial x}\left[\frac{\partial}{\partial t}\int\limits_{-B/2}^{B/2}\int\limits_{z_0}^{H} \overline{w}_H\, d\xi\, dy - 2\int\limits_{z_0}^{H} \overline{w}_{H,B/2}\, d\xi \frac{\partial B}{\partial t} \right] -$$

$$\frac{\partial H}{\partial t}\left[\frac{\partial}{\partial x}\int\limits_{-B/2}^{B/2}\int\limits_{z_0}^{H} \overline{w}_H\, d\xi\, dy - 2\int\limits_{z_0}^{H} \overline{w}_{H,B/2}\, d\xi \frac{\partial B}{\partial x} \right] +$$

$$\int\limits_{-B/2}^{B/2} \frac{\partial}{\partial t}\int\limits_{z_0}^{H} \overline{w}_{z_0} \frac{\partial z_0}{\partial x}\, d\xi\, dy$$

$$= \frac{\partial^2}{\partial x \partial t}\int\limits_F \int\limits_{z_0}^{H} \overline{w}\, d\xi\, dF + \int\limits_{-B/2}^{B/2} \frac{\partial}{\partial t}\int\limits_{z_0}^{H} \overline{w}_{z_0} \frac{\partial z_0}{\partial x}\, d\xi\, dy,$$

$$\int\limits_F R_x\, dF = \int\limits_F \left[\frac{\partial}{\partial x}(\overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'}) - \nu \nabla^2 \overline{u} \right] dF$$

$$= \int\limits_F \frac{\partial}{\partial x}(\overline{u'u'})\, dF + \int\limits_F \left(\frac{\partial}{\partial z}(\overline{u'w'}) - \nu \nabla^2 \overline{u} \right) dF,$$

$$\int\limits_F \frac{\partial \overline{w}^2}{\partial x}\, dF = \frac{\partial}{\partial x}\int\limits_F \overline{w}^2\, dF - \frac{\partial H}{\partial x}\int\limits_{-B/2}^{B/2} \overline{w}_H^2\, dy + \int\limits_{-B/2}^{B/2} \overline{w}_{z_0}^2 \frac{\partial z_0}{\partial x}\, dy,$$

$$\int_F \frac{\partial^2}{\partial x^2}\int_{z_0}^{H} \overline{w}\,\overline{u}\,d\xi\,dF = \frac{\partial^2}{\partial x^2}\int_F\int_{z_0}^{H} \overline{w}\,\overline{u}\,d\xi\,dF -$$

$$2\frac{\partial H}{\partial x}\left[\frac{\partial}{\partial x}\int_{-B/2}^{B/2}\int_{z_0}^{H}(\overline{w}\,\overline{u})_H\,d\xi\,dy - 2\int_{z_0}^{H}(\overline{w}\,\overline{u})_{H,B/2}\,d\xi\frac{\partial B}{\partial x}\right] -$$

$$\frac{\partial^2 H}{\partial x^2}\int_{-B/2}^{B/2}\int_{z_0}^{H}(\overline{w}\,\overline{u})_H d\xi dy +$$

$$2\int_{-B/2}^{B/2}\frac{\partial}{\partial x}\int_{z_0}^{H}(\overline{w}\,\overline{u})_{z_0}\frac{\partial z_0}{\partial x}\,d\xi\,dy + \int_{-B/2}^{B/2}\int_{z_0}^{H}(\overline{w}\,\overline{u})_{z_0}\frac{\partial^2 z_0}{\partial x^2}\,d\xi\,dy$$

$$= \frac{\partial^2}{\partial x^2}\int_F\int_{z_0}^{H}\overline{w}\,\overline{u}\,d\xi\,dF + 2\int_{-B/2}^{B/2}\frac{\partial}{\partial x}\int_{z_0}^{H}(\overline{w}\,\overline{u})_{z_0}\frac{\partial z_0}{\partial x}\,d\xi\,dy +$$

$$\int_{-B/2}^{B/2}\int_{z_0}^{H}(\overline{w}\,\overline{u})_{z_0}\frac{\partial^2 z_0}{\partial x^2}d\xi dy,$$

$$\int_F \frac{\partial}{\partial x}\int_{z_0}^{H} R_z\,d\xi\,dF$$

$$= \int_F \frac{\partial}{\partial x}\left[\int_{z_0}^{H}\left(\frac{\partial(\overline{u'w'})}{\partial x} + \frac{\partial(\overline{w'v'})}{\partial y} + \frac{\partial(\overline{w'w'})}{\partial z} - \nu\nabla^2\overline{w}\right)d\xi\right]dF$$

$$= \int_F \frac{\partial}{\partial x}\left[\int_{z_0}^{H}\left(\frac{\partial(\overline{u'w'})}{\partial x} - \nu\nabla^2\overline{w}\right)d\xi + (\overline{w'w'})_H - (\overline{w'w'})_{z_0}\right]dF$$

$$= \int_F \frac{\partial}{\partial x}\int_{z_0}^{H}\left(\frac{\partial(\overline{u'w'})}{\partial x} - \nu\nabla^2\overline{w}\right)d\xi\,dF + \int_F\frac{\partial}{\partial x}\Big((\overline{w'w'})_H - (\overline{w'w'})_{z_0}\Big)dF.$$

Substituting the values of the above integrals into (8), we obtain

$$i + \sigma\frac{\partial L^{-1}}{\partial x} - \frac{\partial H}{\partial x} = j_1 + j_2 + \frac{1}{g_* F}\Bigg[\frac{\partial}{\partial t}\int\limits_F \overline{u}\,dF - \frac{\partial H}{\partial t}\int\limits_{-B/2}^{B/2}\overline{u}_H dy +$$

$$\frac{\partial}{\partial x}\int\limits_F \overline{u}^2\,dF - \frac{\partial H}{\partial x}\int\limits_{-B/2}^{B/2}\overline{u}_H^2\,dy + \int\limits_{-B/2}^{B/2}\overline{u}_{z_0}^2\frac{\partial z_0}{\partial x}\,dy +$$

$$\int\limits_{-B/2}^{B/2}\Big((\overline{u}\,\overline{w})_H - (\overline{u}\,\overline{w})_{z_0}\Big)\,dy + \frac{\partial^2}{\partial x\partial t}\int\limits_F\int\limits_{z_0}^{H}\overline{w}\,d\xi\,dF +$$

$$2\int\limits_{-B/2}^{B/2}\frac{\partial}{\partial x}\int\limits_{z_0}^{H}(\overline{u}\,\overline{w})_{z_0}\frac{\partial z_0}{\partial x}\,d\xi\,dy + \int\limits_{-B/2}^{B/2}\frac{\partial}{\partial t}\int\limits_{z_0}^{H}\overline{w}_{z_0}\frac{\partial z_0}{\partial x}\,d\xi\,dy +$$

$$\frac{\partial^2}{\partial x^2}\int\limits_F\int\limits_{z_0}^{H}\overline{u}\,\overline{w}\,d\xi\,dF + \int\limits_{-B/2}^{B/2}\int\limits_{z_0}^{H}(\overline{u}\,\overline{w})_{z_0}\frac{\partial^2 z_0}{\partial x^2}\,d\xi\,dy - \frac{\partial}{\partial x}\int\limits_F \overline{w}^2\,dF +$$

$$\frac{\partial H}{\partial x}\int\limits_{-B/2}^{B/2}\overline{w}_H^2\,dy - \int\limits_{-B/2}^{B/2}\overline{w}_{z_0}^2\frac{\partial z_0}{\partial x}\,dy - \int\limits_F\frac{\partial}{\partial x}(\overline{w'w'})_{z_0})\,dF\Bigg], \quad (9)$$

where

$$j_1 = \frac{1}{g_* F}\int\limits_F\left[\frac{\partial}{\partial z}(\overline{u'w'}) - \nu\nabla^2\overline{u} + \frac{\partial}{\partial x}\int\limits_{z_0}^{H}\frac{\partial}{\partial x}(\overline{u'w'} - \nu\nabla^2\overline{w})\,d\xi\right]dF,$$

$$j_2 = \frac{1}{g_* F}\int\limits_F\frac{\partial}{\partial x}\Big(\overline{u'u'} + (\overline{w'w'})_H\Big)\,dF.$$

Here j_1 is the dissipative term and j_2 is the term accounting the momentum related to the velocity oscillations in the flow.

In order to reduce equation (9), we assume that the velocity vector is equal to zero on the wet surface. According to [1, 3], the 12th and 13th terms in the parentheses in equation (9) are high-order infinitesimal quantities and do not affect the water flow motion. After using (5), the 2nd, 4th, and 6th terms are canceled. Now equation (9) can be rewritten in the form

$$i + \sigma \frac{\partial L^{-1}}{\partial x} - \frac{\partial H}{\partial x} = j_1 + j_2 + \frac{1}{g_* F}\left[\frac{\partial}{\partial t}\int\limits_F \overline{u}\, dF + \frac{\partial}{\partial x}\int\limits_F \overline{u}^2\, dF + \right.$$

$$\left. \frac{\partial^2}{\partial x \partial t}\int\limits_F \int\limits_{z_0}^{H} \overline{w}\, d\xi\, dF + \frac{\partial^2}{\partial x^2}\int\limits_F \int\limits_{z_0}^{H} \overline{u}\,\overline{w}\, d\xi\, dF\right], \qquad (10)$$

Let us integrate the continuity equation $\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$ over F:

$$\int\limits_F \left(\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z}\right) dF = \int\limits_F \frac{\partial \overline{u}}{\partial x}\, dF + \int\limits_F \frac{\partial \overline{v}}{\partial y}\, dF + \int\limits_F \frac{\partial \overline{w}}{\partial z}\, dF$$
$$= \int\limits_F \frac{\partial \overline{u}}{\partial x}\, dF + \int\limits_F \frac{\partial \overline{w}}{\partial z}\, dF,$$

The integrals in the right-hand side are transformed as follows:

$$\int\limits_F \frac{\partial \overline{u}}{\partial x}\, dF = \frac{\partial}{\partial x}\int\limits_F \overline{u}\, dF - \frac{\partial H}{\partial x}\int\limits_{-B/2}^{B/2} \overline{u}(x,y,H,t)\, dy + \int\limits_{-B/2}^{B/2} \overline{u}(x,y,z_0,t)\frac{\partial z_0}{\partial x}\, dy$$
$$= \frac{\partial}{\partial x}\int\limits_F \overline{u}\, dF = \frac{\partial}{\partial x}(UF),$$

where $U = \frac{1}{F}\int\limits_F \overline{u}\, dF$ is the average velocity of the flow,

$$\int\limits_F \frac{\partial \overline{w}}{\partial z}\, dF = \int\limits_{-B/2}^{B/2}\int\limits_{z_0}^{H} \frac{\partial \overline{w}}{\partial z}\, d\xi\, dy = \int\limits_{-B/2}^{B/2} (\overline{w}_H - \overline{w}_{z_0})\, dy$$
$$= \int\limits_{-B/2}^{B/2} \overline{w}_H\, dy = \int\limits_{-B/2}^{B/2} \left(\overline{u}_H \frac{\partial H}{\partial x} + \frac{\partial H}{\partial t}\right) dy$$
$$= \frac{\partial H}{\partial x}\int\limits_{-B/2}^{B/2} \overline{u}(x,y,H,t)\, dy + \frac{\partial H}{\partial t}B = B\frac{\partial H}{\partial t} = \frac{\partial F}{\partial t}.$$

Thus,

$$\frac{\partial (UF)}{\partial x} + \frac{\partial F}{\partial t} = 0.$$

To turn to the hydraulic idealization, we need to make some assumptions that would allow us to express all the terms in equation (9) in terms of the variables that characterize the flow as a whole.

We have $\overline{w} = w_1 + w_2$, where w_1 is due to the fact that the velocity vector trajectories are not parallel to each other, and w_2 is due to the movement of these trajectories in space, and with allowance for formula (5), we can write down

$$\overline{u} = \delta U, \qquad w_1 = \varepsilon \overline{u}_H \frac{\partial H}{\partial x}, \qquad w_2 = \eta \frac{\partial H}{\partial t}, \tag{11}$$

where δ, ε, η are some functions of x, y, z and t. Therefore,

$$\overline{w} = \varepsilon \overline{u}_H \frac{\partial H}{\partial x} + \eta \frac{\partial H}{\partial t}, \qquad \overline{u}_H = \delta(x, y, H, t) U,$$

$$\int\limits_F \int\limits_{z_0}^{H} \overline{w}\, d\xi\, dF = \int\limits_F \int\limits_{z_0}^{H} \left(\varepsilon \overline{u}_H \frac{\partial H}{\partial x} + \eta \frac{\partial H}{\partial t} \right) d\xi\, dF = \frac{F^2}{B} \left(\beta_1 U \frac{\partial H}{\partial x} + \beta_2 \frac{\partial H}{\partial t} \right),$$

$$\int\limits_F \overline{u}^2 dF = \int\limits_F \delta^2(x, y, z, t) U^2\, dF = \alpha' F U^2,$$

where

$$\alpha' = \frac{1}{F} \int\limits_F \delta^2(x, y, z, t)\, dF,$$

$$\beta_1 = \frac{B}{F^2} \int\limits_F \delta(x, y, H, t) \int\limits_{z_0}^{H} \varepsilon(x, y, \xi, t)\, d\xi\, dF,$$

$$\beta_2 = \frac{B}{F^2} \int\limits_F \int\limits_{z_0}^{H} \eta(x, y, \xi, t)\, d\xi\, dF.$$

Using the similar transformations gives

$$\int\limits_F \int\limits_{z_0}^{H} \overline{w}\, \overline{u}\, d\xi\, dF = \frac{F^2}{B} \left(\beta_3 U^2 \frac{\partial H}{\partial x} + \beta_4 U \frac{\partial H}{\partial t} \right),$$

where

$$\beta_3 = \frac{B}{F^2} \int\limits_F \delta(x, y, H, t) \int\limits_{z_0}^{H} \varepsilon(x, y, \xi, t) \delta(x, \xi, t)\, d\xi\, dF,$$

$$\beta_4 = \frac{B}{F^2} \int\limits_F \int\limits_{z_0}^{H} \delta(x, y, \xi, t) \eta(x, y, \xi, t)\, d\xi\, dF.$$

In hydraulics, the dissipative term is taken as $j_1 = \frac{Q^2}{K^2}$, where $Q = UF$ is the flow discharge and K is the discharge capacity. It is natural to consider that j_2 may be represented as

$$j_2 = \frac{1}{g_* F}\frac{\partial}{\partial x}(F\alpha'' U^2),$$

where $\alpha'' = \alpha''(x,t)$ is the coefficient of momentum of oscillations.

However, there is a difference between this expression for j_2 and the similar expressions (11): the latter are quite accurate while there are no assumptions regarding the functions δ, ε, η α', β_1, β_2, β_3, β_4. And the former one is a hypothesis regardless of the value of α''. Then we consider that the full coefficient of momentum $\alpha = \alpha' + \alpha''$ is constant, as it is commonly accepted in hydraulics. Using the continuity equation we can easy obtain

$$\begin{aligned}
j_2 &+ \frac{1}{g_* F}\left(\frac{\partial}{\partial t}\int\limits_F \overline{u}\, dF + \frac{\partial}{\partial x}\int\limits_F \overline{u}^2\, dF\right) \\
&= \frac{1}{g_* F}\frac{\partial}{\partial x}(F\alpha'' U^2) + \frac{1}{g_* F}\left(\frac{\partial}{\partial t}(UF) + \frac{\partial}{\partial x}(\alpha' F U^2)\right) \\
&= \frac{1}{g_* F}\left[\alpha\frac{\partial}{\partial x}(FU^2) + \frac{\partial}{\partial t}(UF)\right] \\
&= \frac{1}{g_* F}\left[\alpha\left(\frac{\partial U}{\partial x}FU + U\frac{\partial FU}{\partial x}\right) + \left(F\frac{\partial U}{\partial t} + U\frac{\partial F}{\partial t}\right)\right] \\
&= \frac{1}{g_* F}\left(\alpha F U\frac{\partial U}{\partial x} - \alpha U\frac{\partial F}{\partial t} + F\frac{\partial U}{\partial t} + U\frac{\partial F}{\partial t}\right) \\
&= \frac{1}{g_*}\left(\frac{\partial U}{\partial t} + \alpha U\frac{\partial U}{\partial x} - \frac{\alpha - 1}{F}U\frac{\partial F}{\partial t}\right).
\end{aligned}$$

Using the previous calculations, we can write down equation (10) in the final form

$$\begin{aligned}
i + \sigma\frac{\partial L^{-1}}{\partial x} - \frac{\partial H}{\partial x} = \frac{Q^2}{K^2} &+ \frac{1}{g_*}\left(\frac{\partial U}{\partial t} + \alpha U\frac{\partial U}{\partial x} - \frac{\alpha - 1}{F}U\frac{\partial F}{\partial t}\right) + \\
&\frac{1}{g_* F}\left[\frac{\partial^2}{\partial t\partial x}\frac{F^2}{B}\left(\beta_1 U\frac{\partial H}{\partial x} + \beta_2\frac{\partial H}{\partial t}\right) + \right. \\
&\left.\frac{\partial^2}{\partial x^2}\frac{F^2}{B}\left(\beta_3 U^2\frac{\partial H}{\partial x} + \beta_4 U\frac{\partial H}{\partial t}\right)\right],
\end{aligned} \tag{12}$$

Equation (12) is obtained under the constraint that the flow has a vertical plane of symmetry. This restriction is not essential as equation (12) is also valid for the flows, that do not have a plane of symmetry [1].

Equation (12) is used only to analyze the stability of a steady state flow. According to the analysis of wave motions presented in [1, 3], we can neglect the terms that account for the surface tension $\sigma\dfrac{\partial L^{-1}}{\partial x}$, the influence of the flow curvature and the vertical flow acceleration. In addition, due to the small factor $\alpha - 1$ we can not take into account the last term in the parentheses. Then, we obtain the Saint–Venant equation in the form

$$\begin{aligned} &i - \frac{\partial H}{\partial x} = \frac{Q^2}{K^2} + \frac{1}{g_*}\frac{\partial U}{\partial t} + \frac{\alpha}{g_*}U\frac{\partial U}{\partial x}, \\ &\frac{\partial Q}{\partial x} + \frac{\partial F}{\partial t} = 0, \end{aligned} \tag{13}$$

or, taking into account $U = Q/F$, equation (13) can be rewritten in another form (via discharge):

$$\begin{aligned} &\frac{1}{g_* F}\frac{\partial Q}{\partial t} + \frac{2Q}{g_* F}\frac{\partial Q}{\partial x} + \frac{Q|Q|}{K^2} + \frac{\partial Z}{\partial x} - I = 0, \\ &\frac{\partial Q}{\partial x} + B\frac{\partial Z}{\partial t} = 0. \end{aligned} \tag{14}$$

These Saint–Venant equations are a hyperbolic system of partial differential equations. For the solution of problems with this system we need to establish two initial and two boundary conditions, for example:

$$\begin{aligned} Z(x,0) &= Z^0(x), \qquad Z(0,t) = Z_0(t), \qquad x \in [0,L], \\ Q(x,0) &= Q^0(x), \qquad Z(L,t) = Z_L(t), \qquad t \in [0,T]. \end{aligned} \tag{15}$$

In some cases, for a slowly changing unsteady water motion it is possible with allowance for the error of the input data, not usually exceeding 10^{-3} to use a simplified version of the Saint–Venant equations. It can be obtained by dropping the inertial equation terms, which are within the measurement error accuracy:

$$Q|Q| = K^2\Big(I - \frac{\partial t}{\partial x}\Big), \qquad \frac{\partial Q}{\partial x} + B\frac{\partial Z}{\partial t} = d.$$

This system is of the parabolic type. For this system to be uniquely solvable, we need to add one initial and two boundary conditions.

Thus, mathematical modeling of an unsteady smoothly changing water flow in open channels requires solving the following boundary value problem. We need to find the solution $Z(x,t)$ and $Q(x,t)$ to system (14) in the domain $x \in [0,L]$, $t \in [0,T]$ satisfying conditions (15).

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