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The solution of the one-dimensional unsteady flow problem in the Lena river delta

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1. Introduction

The research into the hydrological regime of the river estuaries conventionally concerns the distribution of the flow across their channels and branches.

There is a known hydraulic method for calculating the hydrological regimes in the river deltas [1]. It is based on a system of the balance equations for branching nodes and uses the hydraulic formulas to describe the free surface level gradient in stationary conditions.

In [2–5] the complex river channel systems are simulated by the numerical solution of the Saint–Venant equation system using a variety of implicit difference schemes. The unsteady flows in the channels are described using a one-dimensional model. At the points of branching, the conjugation conditions are formulated. To solve the arising equations, the authors employ a stable sweeping algorithm which accounts for the tree-like graph structure. Altogether, these constitute the approach to solving the Saint–Venant equation system.

It should be noted that to apply the numerical solution of the Saint–Venant equation system to natural watercourses, one needs to prepare a representative and consistent dataset. This would ensure the accuracy of the solution to be.

This paper considers the mathematical model of the complexly braided Lena estuary. The model describes the hydrological regime of the river starting at the Kusur gauging station and ending at the outlets.

Also, this paper investigates the hydrological regime of the navigable Bykovskaya branch. The digital terrain model required in the equations of motion is obtained using the Google Earth Application.

2. Statement of the problem

We consider the Saint–Venant equations that describe a slowly changing unsteady fluid flow in open channels. If we ignore the effects of the wind, the pressure and the distributed inflow, these equations in the one-dimensional case have the following form:

$$B\frac{\partial Z}{\partial t} + \frac{\partial Q}{\partial x} = 0, \qquad (1)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{\omega}\right) = -g\omega\left(\frac{\partial Z}{\partial x} + \frac{Q|Q|}{K^2}\right), \tag{2}$$

where x is the coordinate taken alongside the channel axis, t is the time, $Q(x,t)$ is the discharge, $Z(x,t)$ is the free surface level, $h(x,t)$ is the flow depth, $B(x,h)$ is the width of the surface when the depth is equal to h, ω is the cross-section area of the flow, $K(x,h)$ is the discharge capacity, and g is the free fall acceleration.

A complexly braided estuary can be described as a system consisting of two types of elements, that is, open channels and conjugation points. This system can be represented as a planar graph with its vertices corresponding to conjugation points and graph edges (segments of the axis x) to open channels.

In the graph, we consider the Saint–Venant equations (1), (2). We choose the discharge $Q(x,t)$ and the free surface level $Z(x,t)$ as unknown distributed parameters, and $Q_*(t)$, $Z_*(t)$—as unknown concentrated parameters.

The initial conditions for the distributed parameters are set on every segment

$$Q(x,0) = Q^0(x), \qquad Z(x,0) = Z^0(x). \tag{3}$$

For the concentrated parameters, the conjugation conditions are set. The latter connect the values of the distributed parameters on the segment edges and the values of the concentrated parameters in the vertices.

Let us enumerate the segments and vertices independent of each other. Suppose that μ^j is a set of the indices of segments that are connected to the jth vertex. Let μ^j_+ and μ^j_- denote the sets of the indices of segments that are connected to the jth vertex by their right and left edges, respectively.

If we consider subcritical flows (the average flow velocity is less than the velocity of small disturbance propagation), the conjugation conditions can be presented in the following forms:

a) the balance of the discharge in a vertex

$$\sum_{i\in\mu^j_+} Q_i - \sum_{i\in\mu^j_-} Q_i = Q^j_*; \tag{4}$$

b) the free surface level coupling

$$Z_i = Z^j_*, \qquad i \in \mu^j; \tag{5}$$

c) if the inflow or the outflow is set in a vertex, a concentrated parameter relation is needed:

$$Q^j_* = f^j(t, Z^j_*); \tag{6}$$

if a concentrated volume is set in a vertex,

$$Q_*^j = -\Omega \frac{\partial Z_*^j}{\partial t}, \tag{6'}$$

where $\Omega = \Omega(Z)$ is the surface area of the concentrated volume.

The boundary conditions in the starting and ending cross-sections of the system are considered as a special case of the conjugation condition. For example, if we set the boundary condition in the form of time dependence, that is, $Q = f(t)$ or $Z = Z(t)$, then relations (6) will have the form $Q_*^j = f(t)$, $Z_*^j = Z(t)$, respectively.

3. The numerical solution method

The problems related to the hydrological regime of complexly braided open channels are best solved using the methods based on the implicit difference schemes. The implicit schemes are absolutely stable with any relative scale of grid parameters and allow us to choose grid steps only from considerations of accuracy.

Let us formulate the system (1), (2) in the characteristic form. This will allow us to represent the difference equations for both the inner and the boundary segment points in a convenient and uniform manner

$$R\frac{\partial U}{\partial t} + \Lambda R \frac{\partial U}{\partial x} = F, \tag{7}$$

where

$$\begin{gathered} U = \begin{pmatrix} Q \\ Z \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & -B(v-c) \\ 1 & -B(v+c) \end{pmatrix}, \\ F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} v+c & 0 \\ 0 & v-c \end{pmatrix}, \\ F_{1,2} = g\omega \left[\mathrm{Fr} \cdot I - \frac{Q|Q|}{K^2} \right]. \end{gathered} \tag{8}$$

Here $\mathrm{Fr} = \frac{v^2}{c^2}$ is the Froude number; $c = \sqrt{\frac{g\omega}{B}}$ is the speed of small disturbance propagation; $v = \frac{Q}{\omega}$ is the cross-section average velocity.

In the implicit difference scheme used, the space and time derivatives are approximated the following way:

$$\left(\frac{\partial U}{\partial t}\right)_n = \frac{U_n^{k+1} - U_n^k}{\tau} + O(\tau),$$
$$\left(\frac{\partial U}{\partial x}\right)_n = \frac{U_{n+1}^{k+1} - U_{n-1}^{k+1}}{2\Delta} + O(\Delta^2), \quad \text{for } n = 1, \ldots, N-1,$$
$$\left(\frac{\partial U}{\partial x}\right)_0 = \frac{U_1^{k+1} - U_0^{k+1}}{\Delta} + O(\Delta),$$
$$\left(\frac{\partial U}{\partial x}\right)_N = \frac{U_N^{k+1} - U_{N-1}^{k+1}}{\Delta} + O(\Delta).$$

The coefficients in the difference equations are calculated based on the known solution for $t = t_k$. The right-hand side can be linearized and written in the following form:

$$F_n^{k+1} = F_n^k + \left(\frac{\partial F}{\partial U}\right)_n^k (U_n^{k+1} - U_n^k).$$

At the boundary points, where $n = 0$ and $n = N$, we use only the difference relations that correspond to the characteristic directions. When the second equation from (7) is approximated, that is, $n = 0$, we use

$$\frac{\partial x}{\partial t} = v - c.$$

And when the first equation from (7) is approximated, that is, $n = N$, we employ

$$\frac{\partial x}{\partial t} = v + c.$$

Considering the above relations, the following system of difference equations can be obtained for each segment:

$$A_n \left(U_{n+1}^{k+1} - U_{n-1}^{k+1}\right) + B_n U_n^{k+1} = D_n, \qquad (9)$$
$$n = 1, 2, \ldots, N-1,$$

where

$$A_n = \frac{\tau}{2\Delta}(\Lambda R)_n^k, \quad B_n = R_n^k - \tau \left(\frac{\partial F}{\partial U}\right)_n^k, \quad D_n = B_n^k U_n^k + \tau F_n^k. \qquad (10)$$

For the boundary points $n = 0$, $n = N$, the difference equations have the following forms, respectively:

$$2A_0^{(2)} U_1^{k+1} + \left(B_0^{(2)} - 2A_0^{(2)}\right) U_0^{k+1} = D_0^{(2)}, \qquad (11)$$
$$\left(B_N^{(1)} + 2A_N^{(1)}\right) U_N^{k+1} - 2A_N^{(1)} U_{N-1}^{k+1} = D_N^{(1)}, \qquad (12)$$

where indices (1) and (2) mean that only the first or the second matrix row is taken.

The system of equations (9), (11), (12) is not closed. In order to close it we need the boundary conditions or the conjugation conditions that are linearized as needed:

- if $Q = f(Z)$ is set, then

$$Q_*^{k+1} = f(Z_*^k) + \left(\frac{\partial f}{\partial Z}\right)^k \left(Z_*^{k+1} - Z_*^k\right); \tag{13}$$

- if concentrated volume is considered, then

$$Q_*^{k+1} = -\Omega(Z_*^k) \left(\frac{Z_*^{k+1} - Z_*^k}{\tau}\right). \tag{14}$$

Solving the system of equations (9), (11), (12), (4)–(6), which are not necessarily linear, can be computationally complex without taking the structure of the equations into account. The paper [5] offers a variation of the sweep method that takes advantage of the tridiagonal matrix structure and the structure of a system of open channels.

4. Investigation of numerical solution convergence to an exact solution

With some restrictions on the channel parameters, the Saint–Venant equation system can be reduced to the parabolic diffusion-convection partial differential equation (PDE).

If we ignore Q_t and $\left(\frac{Q^2}{\omega}\right)_x$ in (2) and assume that the width of an open channel is constant, then it is possible, with some transformations, to derive a second order PDE only with the dependent variable $Q(x, t)$:

$$Q_t - \frac{K^2}{2B|Q|} Q_{xx} + \frac{K_h Q}{BK} Q_x = 0. \tag{15}$$

Equation (15) represents the convection-diffusion equation with the diffusion coefficient $a = \frac{K^2}{2B|Q|}$ and the convection speed $b = \frac{K_h Q}{BK}$ for the discharge. With a and b being constant, the analytical solution can be obtained with the help of the Fourier method (also known as separation of variables). The solution satisfies the initial and boundary conditions for the discharge in the open channel:

$$Q(x,0) = Q_0(x), \quad Q(0,t) = Q_1(t), \quad Q(L,t) = Q_2(t). \tag{16}$$

Furthermore, using the found solution and equation (1) it is possible to deduce the depth of the flow.

In order to investigate the convergence and check the correctness of the algorithm, the analytical solution is compared to the numerical one. For this we repeat the experiment conducted in [6] which considers a part of the Kuban mountain river. Its average bed inclination is assumed to be equal to $1,8 \cdot 10^{-3}$, which allows one to describe the movement of a flood wave with (15).

The following dependencies are taken as the given data:

$$\begin{aligned} Q_0(x) &= 4(6.0606 \cdot 10^{-10}x^2 - 2.739 \cdot 10^{-4}x + 100), \\ Q_1(t) &= 4(2.0552 \cdot 10^{-15}t^3 - 7.663 \cdot 10^{-10}t^2 + 1.4 \cdot 10^{-4}t + 100), \\ Q_2(t) &= 4(1.956 \cdot 10^{-15}t^3 - 9.285 \cdot 10^{-10}t^2 + 2.294 \cdot 10^{-4}t + 80.2581). \end{aligned} \tag{17}$$

The considered segment has the following characteristics: the length $L = 90$ km, the width $B = 70$ m, the initial depth $h_0 = 2.5$ m, the roughness code $n = 0.04$. Its shape is to be considered rectangular. The integration period is 5 days. The grid steps are $\Delta x = 0.5$ km and $\Delta t = 1800$ s. We also set $a = 11635$ and $b = 3.5759$. These values are obtained by averaging their values in the numerical solution over time.

Figure 1 depicts the exact solution of problem (15), (17). Figure 2 presents the results of the comparison of the numerical and analytical solutions. For the beginning, the dependence of the discharge on time is presented then subsequently the middle and the end of the segment. The

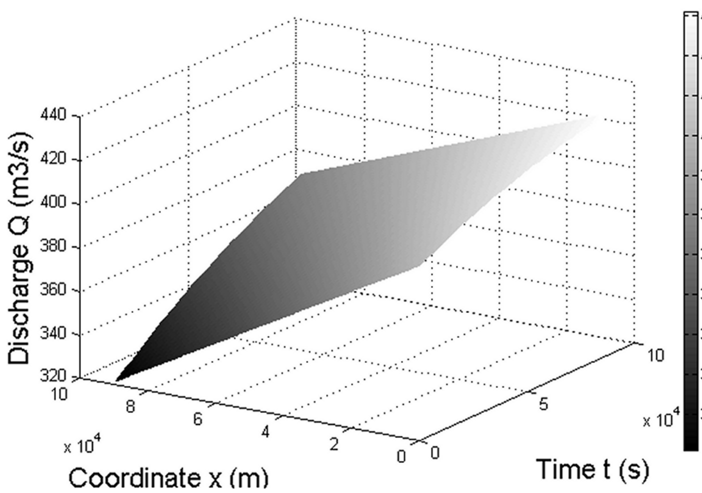


Figure 1. The analytical solution

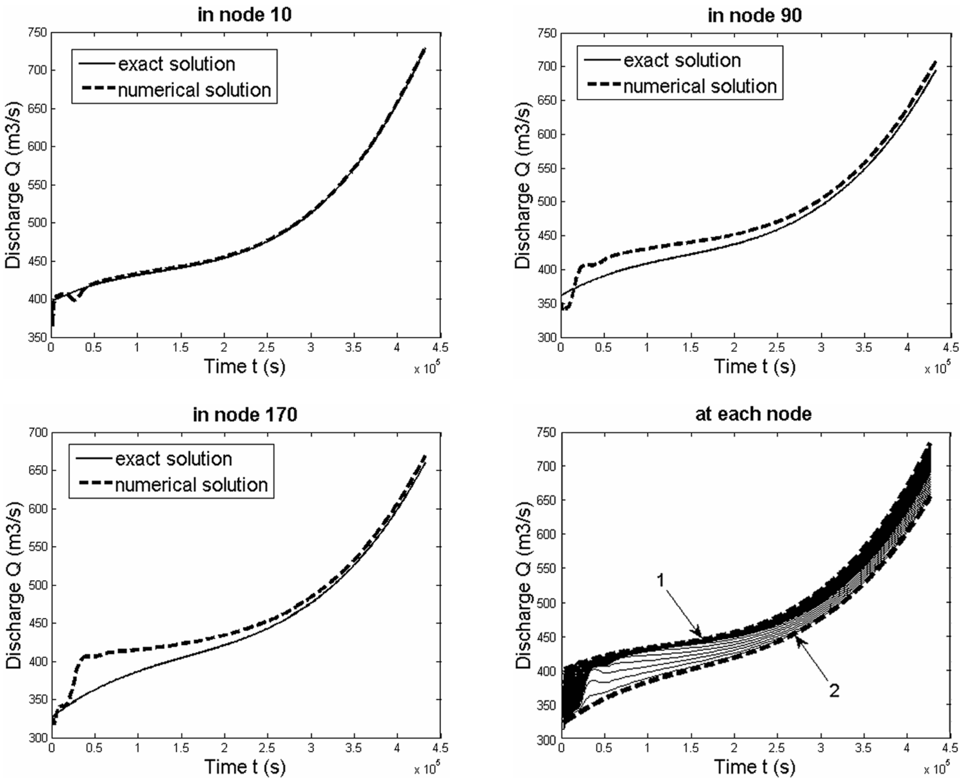


Figure 2. A comparison of the numerical and analytical solutions

last plot shows the numerical solution at every point of the segment. There, the bold dashed lines indicate to the discharge in the segment ends, and the solid lines correspond to the discharge at the inner points.

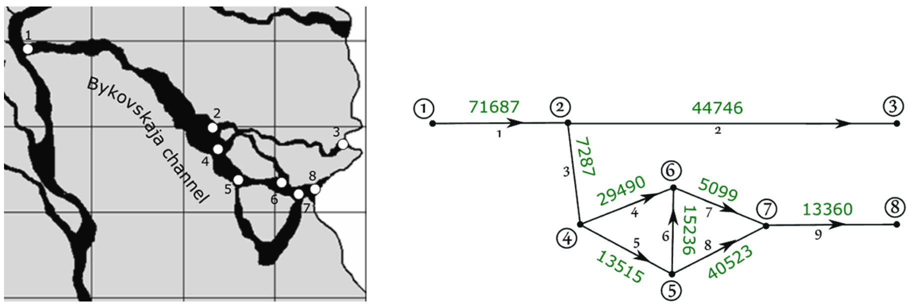
The results obtained demonstrate the convergence of the numerical solution to the exact one for a chosen hydrological regime.

5. The numerical modeling of an unsteady water flow in the Bykovskaya branch of the Lena river delta

As a real object of our research, one of the mainstream Lena channels is chosen, namely, the Bykovskaya branch. It is currently the primary navigable waterway of Lena river delta.

Figure 3 depicts the most affluent channels of this branch schematically represented by a planar graph. The latter consists of 9 computational segments. The number near every segment denotes its length.

The Bykovskaya branch is situated in the eastern part of the delta. It is directed south-eastward and flows into the Laptev sea near the Bykov cape



a b

Figure 3. The Bykovskaya branch: a — the primary channels, b — the computational graph depicting the segment lengths (m)

(see the 7th graph vertex in Figure 3a). Its length is 105 km and its sinuosity coefficient is 1.14. Throughout its way, a large number of secondary channels flow in and out of it. None of them are navigable except for the Sinitsyn channel (the first half of the segment connecting the 5th and the 7th graph vertices) and an unnamed channel near the Bykov cape.

It is assumed that the cross-sections of all channels in the branch are triangular. Its width $B(x,t)$ is considered to be a function of time and space. The initial width was measured using the *Google Earth* program. The time step is equal to 1800 s and the x-coordinate step being equal to 500 m on average.

The initial free surface level is assumed to be zero. The initial depth is calculated using the hydrologic-morphometric formula proposed in [7]:

$$h = 0.42B^{1/3}.$$

Figure 4 depicts a schematized profile of the channel bed obtained with the formula $Z_0 = Z - h$.

Due to the fact that the initial discharge is not in advance known, the simulation is run in two stages. The first stage consists in running the simulation with the constant input discharge and the constant output levels for a specified period of time. Namely, we set the input discharge at vertex 1 to 5,000 m^3/s, the levels at vertices 3 and 8 are set to zero and the simulation runs for 2 days. The initial discharge and level are set to zero. The chosen period of time allows the process to reach a nearly-steady state. Figure 5 displays the distribution of the discharge across the branch individual channels.

The results obtained are used for setting the initial state at the second stage which simulates the unsteady flow in the branch.

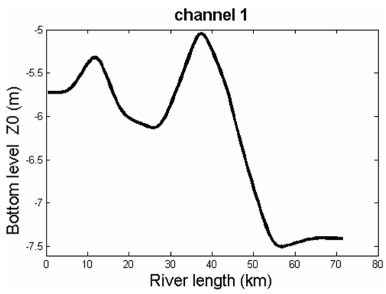


Figure 4. The schematized bed profile

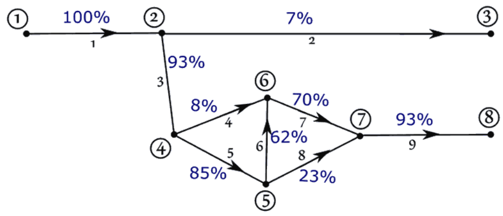


Figure 5. The distribution of the discharge across the channels

The unsteady water flow in the Lena river delta is the result of two factors. The first one is the runoff caused by rainfall and the second one is the wind-induced water level fluctuations. In this paper, we account for the first factor only. Therefore, the discharge at vertex 1 is a function of time $Q = f(t)$ and the levels at vertices 3 and 8 are zero.

To define $f(t)$, we use the data collected at the Kusur gauging station in 2008. The Kusur gauging station is located upstream of the Stolb island, which is the main branching point of the delta. In [8], the distribution of water flow among the delta parts is discussed over a period of 1953–1990. Therein, the Bykovskaya branch is $\approx 25\,\%$ of the total flow.

We have simulated the main navigable period, that is, from the beginning of May up to the end of October. The time dependence of the discharge in vertex 1 is illustrated in Figure 6.

After imposing all boundary conditions, the problem can be solved. The results of the modeling the unsteady flow in the Bykovskaya branch are depicted in Figures 7, 8.

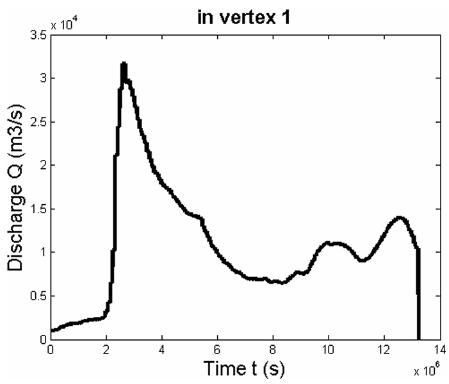


Figure 6. The discharge boundary condition in vertex 1

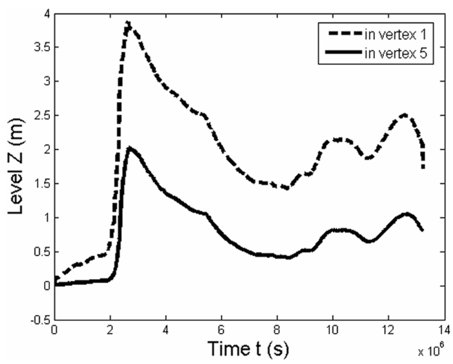


Figure 7. The free surface levels in vertices 1 and 5

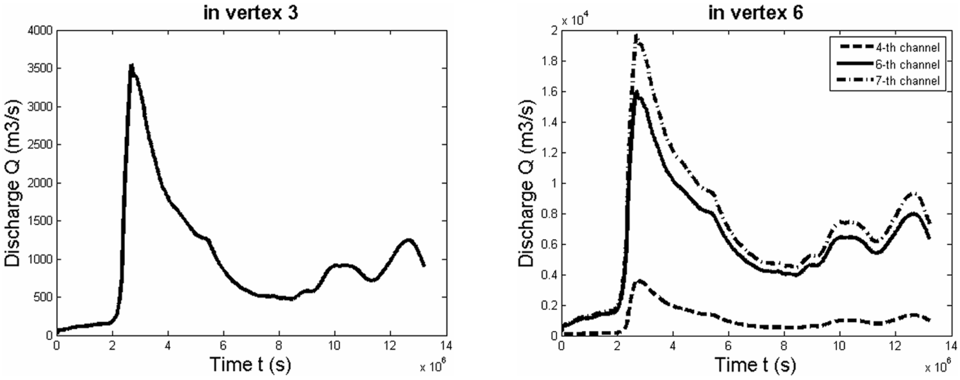


Figure 8. The discharge in selected vertices

6. Conclusion

This paper, among other objectives, aims at validation of the numerical solution to the Saint–Venant equations. With some assumptions, the latter can be reduced to the convection–diffusion equation, for which an analytical solution can be easily built. In this paper, a comparison of these solutions is presented. Moreover, the paper considers modeling the free surface level and the discharge in the Bykovskaya branch of the Lena river delta. which is one of the primary navigable watercourses in the delta. The numerical solution allows us to obtain the qualitative depiction of the water movement in the delta. The width of the channels is obtained from the Google Earth Application, and the discharge data come from the measurements collected at the Kusur gauging station in 2008.

The numerical simulation based of the hydrological regime in the delta allows us to obtain the characteristics of the flow that are hard to measure in reality.

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