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Application of a mixed finite element method for solving 2D nonlinear vorticity equation

V.I. Kuzin, V.V. Kravtchenko

Abstract. A new two-step scheme has been obtained as a result of application of the finite element method (FEM) and the splitting up method to the 2D nonlinear vorticity equation. At the first step, the conforming piecewise-linear finite elements are used; at the second step, non-conforming ones are made use of. The efficiency of the scheme was tested on each splitting step separately as well as on the problem as a whole.

Introduction

The problem of finding a plane non-stationary circulation is one of typical among the ocean dynamics problems. This initial-boundary value problem is described by a 2D nonlinear vorticity equation. In this paper, we present a scheme for which the splitting combined with a finite element method (FEM) is used. In this case, splitting is carried out at different steps of constructing a numerical model, including both splitting in terms of physical processes allowing linearization of the initial problem and further splitting with respect to time of one of the FEM operators obtained. For constructing FEM operators at the steps of splitting in terms of physical processes, different types of finite elements are used. Hence, it appears possible to essentially reduce the number of grid points in a numerical scheme when passing from one splitting step to another. At the first step, for solving the linear stream function equation, conforming piecewise-linear finite elements are used. At the second step, corresponding to the vorticity advection and diffusion, non-conforming finite elements are used. Finite elements of such a type were introduced by M. Crouzeix and P.A. Raviart [1] for solving the stationary Stokes equations. Later these elements were used by B.-L. Hua and F. Tomasset [2] to obtain a noise-free scheme for two-layer shallow water equations. Some of their advantages are listed below. Because of their orthogonality we can avoid the lumping procedure at the splitting steps with respect to time. Also, in comparison with the case of conforming finite elements, a smaller number of grid points is used in the FEM scheme obtained. At the same time, on a standard grid, the degrees of freedom in this case increase by the factor of 3 that may improve the accuracy of an FEM solution. The equations obtained conserve transformation laws for some integral characteristics such as mass and energy with respect to time, which is important for obtaining correct solutions in terms of physical features.

Analysis of the FEM operator obtained after using non-conforming finite elements shows that it can be split into two three-point positive semi-definite operators. As compared to the coordinate-wise splitting, the factorization goes along the broken lines connecting mesh points. Apparently, such an approach has not been used yet in the splitting up theory. On the contrary, in the case of conforming finite elements, there was a splitting into, at least, three operators, including the diagonal direction [3].

An essential problem of using non-conforming finite elements is that an FEM solution does not belong to the space of solvability of the initial problem that demands an additional formal foundation.

1. Statement of the problem

In the domain $Q = \Omega \times (0,T)$, let us consider a 2D nonlinear vorticity equation in terms of a stream function with initial and boundary conditions. In the dimensionless form [4], we have:

$$\begin{aligned} &\Delta\Psi_t + \delta J(\Delta\Psi, \Psi) + \beta \frac{\partial \Psi}{\partial x} + \varepsilon\Delta\Psi - \mu\Delta\Delta\Psi = f, \quad (x,y,t) \in Q; \\ &\Psi(x,y,0) = \Psi^0(x,y), \quad \Psi|_{\partial\Omega} = 0, \quad \Delta\Psi|_{\partial\Omega} = 0. \end{aligned} \tag{1}$$

Here Ω is a bounded uni-connected domain in R^2 space with the boundary $\partial\Omega \in C^2$, $\beta = 1$, $\delta, \varepsilon, \mu \in [\gamma_1, \gamma_2]$, $\gamma_2 \geq \gamma_1 > 0$; $\Psi^0(x,y) \in C^2(\Omega)$, $f = f(x,y,t) \in L_2(\Omega) \times C^0((0,T])$;

$$J(u,v) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} \quad \text{is Jacobian.}$$

In terms of vorticity $\zeta = \Delta\Psi$, equation (1) can be rewritten in the form

$$\begin{aligned} &\zeta_t + \delta J(\zeta, \Psi) + \beta \frac{\partial \Psi}{\partial x} + \varepsilon\zeta - \mu\Delta\zeta = f, \quad \Delta\Psi = \zeta, \quad (x,y,t) \in Q; \\ &\zeta(x,y,0) = \Delta\Psi^0(x,y), \quad \Psi|_{\partial\Omega} = 0, \quad \zeta|_{\partial\Omega} = 0. \end{aligned} \tag{2}$$

Let us divide the interval $[0,T]$ into N_t subintervals with a length τ. According to a weak approximation method, problem (2) will be solved by splitting in terms of physical processes [5]: for $t \in [t_n, t_{n+1}]$,

$$\Delta(\Psi_1)_t + \varepsilon\Delta\Psi_1 + \beta \frac{\partial \Psi_1}{\partial x} = f, \quad \Delta\Psi_1|_{t=t_n} = \zeta_2|_{t=t_n}, \; \Psi_1|_{\partial\Omega} = 0; \tag{3}$$

$$(\zeta_2)_t + \delta J(\zeta_2, \Psi_1^{n+1}) - \mu\Delta\zeta_2 = 0, \quad \zeta_2|_{t=t_n} = \Delta\Psi_1|_{t=t_{n+1}}, \; \zeta_2|_{\partial\Omega} = 0, \tag{4}$$

$n = 0, \dots, N_t - 1$.

Here the first step is in solving the linear stream function equation with forcing, and the second one describes advection and diffusion of the vorticity.

2. Construction of schemes

In Ω, we construct a rectangular grid. Then rectangles of the grid are divided into triangles by diagonals with variable directions, either positive or negative (Figure 1).

Let us consider two types of finite elements:

1. Conforming elements ω^{c}_{pq} are piecewise-linear functions determined by values at vertices of triangles in the following way:

$$\omega^{\mathrm{c}}_{pq}(x_k, y_l) = \begin{cases} 1, & (k,l) = (p,q); \\ 0, & (k,l) \neq (p,q). \end{cases}$$

   Here (x_k, y_l) is a vertex of some triangle of the grid.

2. Nonconforming elements $\omega^{\mathrm{nc}}_{ij}$ are piecewise-linear functions determined by values at midpoints of the sides of triangles in the following way:

$$\omega^{\mathrm{nc}}_{ij}(x_k, y_l) = \begin{cases} 1, & (k,l) = (i,j); \\ 0, & (k,l) \neq (i,j). \end{cases}$$

   Here $(x_k, y_l)$is a midpoint of a side of some triangle of the grid.

Figure 2 presents a view of the functions. Such functions were considered in [2] and [6].

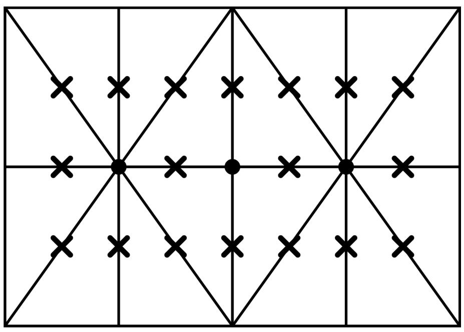

Figure 1. A fragment of the grid. Conforming finite elements are associated with •-nodes and nonconforming ones—with ×-nodes

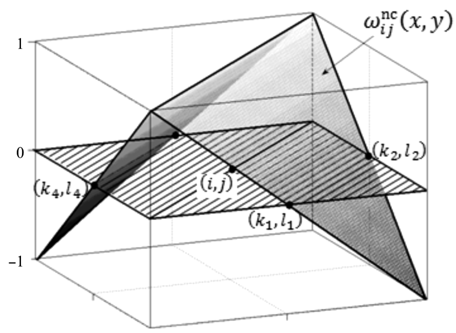

Figure 2. Nonconforming basis function $\omega^{\mathrm{nc}}_{ij}(x, y)$

These functions possess some significant features: $\omega^{\mathrm{nc}}_{ij}$ are orthogonal and each conforming element ω^{c}_{pq} is the half-sum of the non-conforming ones surrounding it.

We will search for a stream function as a linear combination of conforming finite elements and for vorticity as a linear combination of nonconforming ones:

$$\begin{aligned} \Psi_1 \approx \psi^{N^{\mathrm{c}}} &= \sum_{(p,q)\in N^{\mathrm{c}}} \psi_{pq}\, \omega^{\mathrm{c}}_{pq}(x,y), \\ \zeta_2 \approx \varphi^{N^{\mathrm{nc}}} &= \sum_{(i,j)\in N^{\mathrm{nc}}} \varphi_{ij}(t)\, \omega^{\mathrm{nc}}_{ij}(x,y), \end{aligned} \tag{5}$$

where N^{nc} is a set of mesh points including midpoints of the sides of triangles and N^{c} is a set of mesh points including the triangles vertices; $\varphi_{ij}(t)$ and ψ_{pq} are weight coefficients to be determined.

Let us consider splitting steps (3) and (4) in some detail.

After discretization the problem (3) with respect to time we have:

$$\Big(\frac{1}{\tau}+\varepsilon\Big)\Delta\Psi_1^{n+1}+\beta\frac{\partial\Psi_1^{n+1}}{\partial x}=f+\frac{1}{\tau}\Delta\Psi_1^{n},$$

or, taking initial conditions into account,

$$\Big(\frac{1}{\tau}+\varepsilon\Big)\Delta\Psi_1^{n+1}+\beta\frac{\partial\Psi_1^{n+1}}{\partial x}=f+\frac{1}{\tau}\zeta_2^{n}. \tag{6}$$

After application of the Galerkin method to problem (6), a system of linear algebraic equations is obtained:

$$I_1\big(\psi^{N^{\mathrm{c}}},\omega^{\mathrm{c}}_{pq}\big)=-\big(f,\,\omega^{\mathrm{c}}_{pq}\big)-\frac{1}{\tau}\big(\varphi^{N^{\mathrm{nc}}},\,\omega^{\mathrm{c}}_{pq}\big),\quad (p,q)\in N^{\mathrm{c}}, \tag{7}$$

where

$$I_1(u,v)=\int_\Omega\Big(\Big(\frac{1}{\tau}+\varepsilon\Big)u_xv_x+\Big(\frac{1}{\tau}+\varepsilon\Big)u_yv_y-\beta u_xv\Big)\,d\Omega,$$

$\varphi^{N^{\mathrm{nc}}}$ is a nonconforming FEM solution to problem (4) from the previous time step.

System (7) is solved by an iterative technique.

A weak formulation of the problem of the second splitting step (4) is the following:

$$\begin{aligned} &((\zeta_2)_t,\,v)+I_2(\zeta_2,v)=(0,\,v)=0\quad \forall v\in \overset{\circ}{W}{}^1_2(\Omega),\quad t\in(t_n,t_{n+1}], \\ &(\zeta_2(x,y,t_n),\,v)=(\Delta\psi^{N^{\mathrm{c}}},\,v)\quad \forall v\in \overset{\circ}{W}{}^1_2(\Omega), \end{aligned} \tag{8}$$

where $\psi^{N^{\mathrm{c}}}$ is a conforming FEM solution of the first step (3).

Here $\overset{\circ}{W}{}^1_2(\Omega)$ is a subspace of $W^1_2(\Omega)$, which includes the functions vanishing at the boundary of the domain Ω;

$$I_2(u,v) = \int_\Omega \Big(\mu u_x v_x + \mu u_y v_y + \delta \psi^{N^c} u_y v_x - \delta \psi^{N^c} u_x v_y \Big)\, d\Omega.$$

There arises a problem when using the Bubnov–Galerkin method for the search for an FEM solution $\varphi^{N^{nc}}$, which is a linear combination of non-conforming elements. The point is that when defining a weak solution, it is required to carry out the integral relation $\forall v \in \overset{\circ}{W}{}_2^1(\Omega)$ but the functions $\omega_{ij}^{nc}(x,y)$ have discontinuities at the boundaries of their supporter. So, we need to introduce the approximate bilinear form

$$I_2^h(u,v) = \sum_k \int_{T^k} \Big(\mu u_x v_x + \mu u_y v_y + \delta \psi^{N^c} u_y v_x - \delta \psi^{N^c} u_x v_y \Big)\, d\Omega,$$

where T^k are triangles of the domain Ω. In this case a non-conforming FEM solution $\varphi^{N^{nc}}$ does not belong to the required space $\overset{\circ}{W}{}_2^1(\Omega)$ as well. However, the efficiency of the scheme presented was verified earlier with the help of numerical experiments [7].

As a result, with allowance for orthogonality of non-conforming finite elements and their relation with the conforming ones, the following differential equation system is obtained:

$$M^h(\Phi)_t + \Lambda^h \Phi = f, \quad t \in [t_n, t_{n+1}]; \qquad \Phi|_{t=t_n} = X\overline{\Psi}. \tag{9}$$

Here $M^h = \mathrm{diag}(\theta_{ij}/3)$; $[\Lambda^h \Phi]_{ij} = I_2^h(\varphi^{N^{nc}}, \omega_{ij}^{nc})$; $[\Phi]_{ij} = \varphi_{ij}$, $[\overline{\Psi}]_{ij} = \psi_{ij}$, X is a transition matrix from the stream function to the vorticity, obtained from the relation between non-conforming and conforming elements, θ_{ij} is an area of the support of the function ω_{ij}^{nc}

In this case $\Lambda^h = S^h + K^h$, where the first one is a symmetric operator and the second one is a skew-symmetric operator corresponding to the symmetric and skew-symmetric parts of the integral operator.

Analysis of the operator Λ^h shows that it can be presented as sum of two 1D positive semi-definite operators Λ_1^h and Λ_2^h acting along the broken lines connecting non-conforming mesh points (Figure 3). Moreover,

$$\Lambda_1^h = S_1^h + K_1^h; \qquad \Lambda_2^h = S_2^h + K_2^h,$$

where $S^h = S_1^h + S_2^h$, $K^h = K_1^h + K_2^h$, S_r^h are symmetric operators, K_r^h are skew-symmetric ones $(r = 1, 2)$.

Such a decomposition of the grid operator allows the use of the splitting method with respect to time for solving problem (9). In this case, a two-cycle splitting method is used [8].

Let us divide the interval $[t_n, t_{n+1}]$ into the subintervals $t_n + m\tau_1 \le t \le t_n + (m+1)\tau_1$, $\tau_1 = \frac{\tau}{N_1}$, $m = 0, \ldots, N_1 - 1$, N_1 is the number of additional time subintervals. The system of grid equations consists of a sequence of the Crank–Nicholson schemes for the operators Λ_1^h and Λ_2^h constructed on the sub-interval $[t_n + m\tau_1, t_n + (m+1)\tau_1]$:

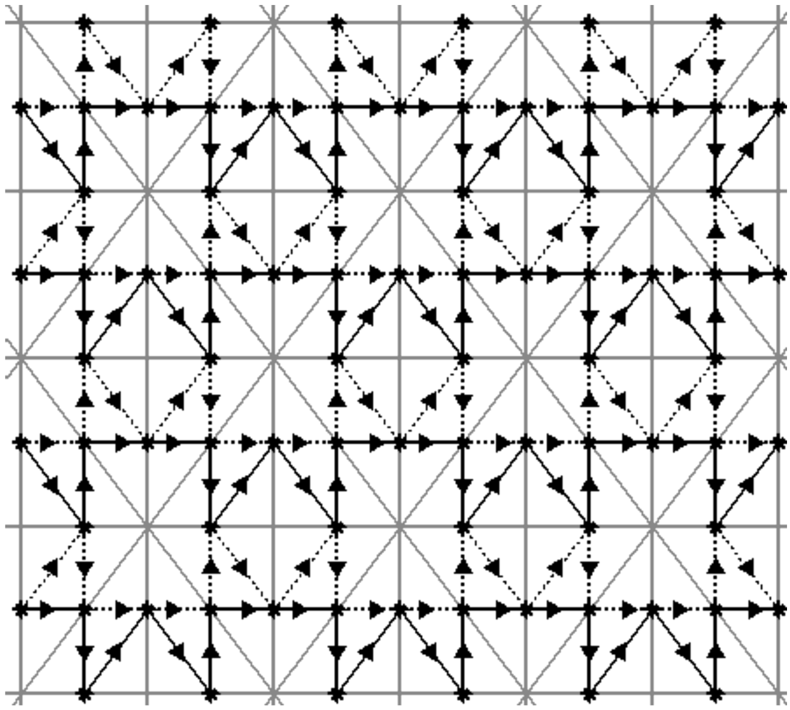

Figure 3. Direction of operators Λ_1^h (—) and Λ_2^h $(\cdots)$

$$\Big(M^h + \frac{\tau_1}{4}\Lambda_1^h\Big)\Phi^{m+1/4} = \Big(M^h - \frac{\tau_1}{4}\Lambda_1^h\Big)\Phi^{m};$$
$$\Big(M^h + \frac{\tau_1}{4}\Lambda_2^h\Big)\Big(\Phi^{m+1/2} - \frac{\tau_1}{2}(M^h)^{-1}f^{m+1/2}\Big) = \Big(M^h - \frac{\tau_1}{4}\Lambda_2^h\Big)\Phi^{m+1/4};$$
$$\Big(M^h + \frac{\tau_1}{4}\Lambda_2^h\Big)\Phi^{m+3/4} = \Big(M^h - \frac{\tau_1}{4}\Lambda_2^h\Big)\Big(\Phi^{m+1/2} + \frac{\tau_1}{2}(M^h)^{-1}f^{m+1/2}\Big);$$
$$\Big(M^h + \frac{\tau_1}{4}\Lambda_1^h\Big)\Phi^{m+1} = \Big(M^h - \frac{\tau_1}{4}\Lambda_1^h\Big)\Phi^{m+3/4}. \qquad (10)$$

To prove approximation of the scheme with respect to time, the Taylor expansion in series with a restriction on the time space (11) is used:

$$\frac{\tau_1}{4}\big\|(M^h)^{-1}\Lambda_r^h\big\| \le 1. \qquad (11)$$

In [8], this method is analyzed. There is shown that the method is absolutely stable.

3. Numerical experiments

The efficiency of the scheme was verified for each splitting steps separately. Some of the tests are listed below.

1. Testing the second splitting step (4):

$$(\zeta_2)_t + \delta J(\zeta_2, \Psi_1) - \mu\Delta\zeta_2 = 0;$$
$$T = 0.1, \quad \delta = 10^{-3}, \quad \mu = 10^{-4}, \quad \Psi_1 = y.$$

The exact solution is

$$\zeta_2 = \frac{1}{\mu\pi}\sin\frac{\pi^2 y}{b}\cdot e^{-\pi^2 t}\cdot\left(1 - pe^{a_1 x} - qe^{a_2 x}\right),$$
$$q = \frac{e^{a_1/2} - 1}{e^{a_1/2} - e^{a_2/2}}, \quad p = 1 - q,$$
$$a_1 = \frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \pi^2(1-\mu)}, \qquad a_2 = \frac{\delta}{2} - \sqrt{\frac{\delta^2}{4} + \pi^2(1-\mu)}.$$

A relative error

$$\mathrm{e_{rel}} = \frac{\max\limits_{(i,j)\in N^{\mathrm{nc}}} |\zeta_2(x_i, y_j, T) - (\varphi_{ij})^{N_t}|}{\max\limits_{(i,j)\in N^{\mathrm{nc}}} |\zeta_2(x_i, y_j, T)|}$$

is shown in Table 1.

Table 1. A relative error with respect to the number of subintervals in space and time

$N_x \times N_y \times N_t$	$\mathrm{e_{rel}}$	$N_x \times N_y \times N_t$	$\mathrm{e_{rel}}$
$10 \times 10 \times 2$	$1.7 \cdot 10^{-2}$	$10 \times 10 \times 20$	$1.8 \cdot 10^{-3}$
$10 \times 10 \times 4$	$4.0 \cdot 10^{-3}$	$20 \times 20 \times 20$	$9.6 \cdot 10^{-4}$

2. A problem with a physical meaning that is in the search for a solution with allowance for real parameters including the boundary layer. This is a problem on stabilization with a constantly acting force $f = 0.1$. Two versions of initial conditions are considered: zero value and solution to the Stommel problem

$$\varepsilon\Delta\Psi + \beta\frac{\partial\Psi}{\partial x} = f\sin\frac{\pi y}{b}.$$

Since both versions with equal parameters differ only in the form of a resulting function in the beginning of the process and by the moment $T = 500$ give solutions of the same kind with a tendency to stabilization, below only the first one is presented.

Table 2. Velocity of stabilization and a maximum of FEM solution with respect to the number of time steps N_t

$N_x \times N_y \times N_t$	$\mathrm{e}_{\mathrm{rel}}$	MAX
$100 \times 100 \times 10$	$9.98 \cdot 10^{-2}$	$1.97 \cdot 10^{-3}$
$100 \times 100 \times 100$	$9.87 \cdot 10^{-3}$	$1.93 \cdot 10^{-2}$
$100 \times 100 \times 1000$	$6.30 \cdot 10^{-4}$	$1.22 \cdot 10^{-1}$
$100 \times 100 \times 10000$	$7.10 \cdot 10^{-6}$	$9.39 \cdot 10^{-2}$
$100 \times 100 \times 100000$	$6.21 \cdot 10^{-8}$	$9.35 \cdot 10^{-2}$

A relative error and a maximum of the FEM solution,

$$\mathrm{e}_{\mathrm{rel}} = \frac{\max\limits_{(p,q)\in N^{\mathrm{c}}} |(\psi_{pq})^{N_t-1} - (\psi_{pq})^{N_t}|}{\mathrm{MAX}}, \qquad \mathrm{MAX} = \max_{(p,q)\in N^{\mathrm{c}}} |(\psi_{pq})^{N_t}|,$$

and its form are results of the test with the following parameters $\delta = 10^{-3}$, $\varepsilon = 10^{-2}$, $\mu = 10^{-4}$ and with a fixed time step τ (Table 2 and Figure 4).

At a fixed moment T, the results for schemes with different time steps τ coincide with one another with an exception of a minor difference in maxima of solutions, which is also in favor of convergence of the method. It is worth to note that the kind of the result corresponds quite well to the current numerical solution of the problem in question.

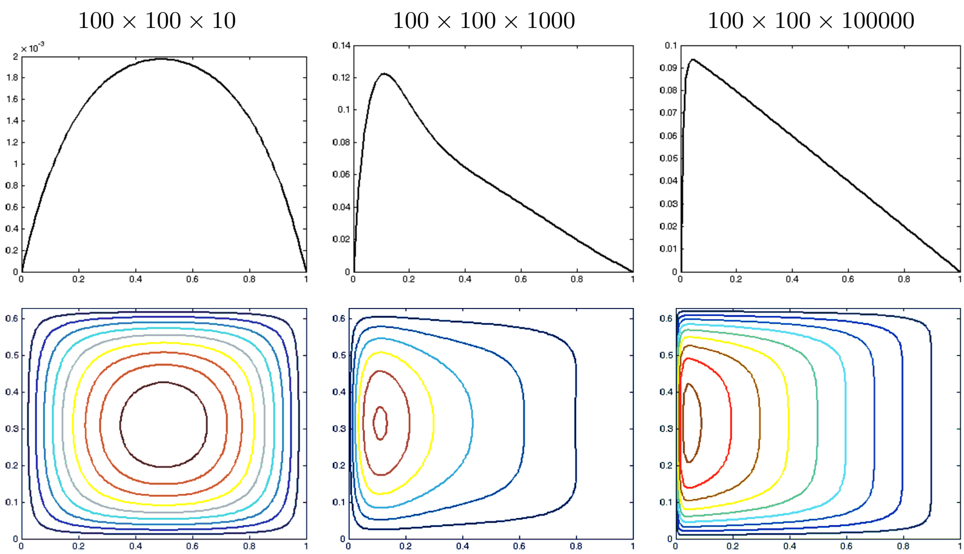

Figure 4. Stabilization of the FEM solution with respect to the number of time steps $100 \times 100 \times N_t$: a section along x with $y = b/2$ (top) and isolines (bottom)

Conclusion

A new two-step scheme has been obtained as a result of application of the finite element method (FEM) and the splitting up method to the 2D nonlinear vorticity equation. At the first step, the conforming piecewise-linear finite elements are used; at the second step, non-conforming ones are made use of. Such elements allow the reduction of the number of grid points in a numerical scheme and presentation of a grid operator as two 1D positive semi-definite operators thus reducing the time needed for successive splitting with respect to time. The efficiency of the scheme was tested on each splitting step separately as well as on the problem as a whole.

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