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Relationships between the characteristics of mutually orthogonal families of curves and surfaces

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Abstract. In the Euclidean space E^3, we consider the family $\{L_\tau\}$ of the curves L_τ with the tangent unit vector $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$ and the family $\{S_\tau\}$ of the surfaces S_τ with the unit normal $\boldsymbol{\tau}$ which are orthogonal to the curves L_τ, i.e., to the field $\boldsymbol{\tau}$. Each of these families continuously fills in a domain D in E^3. We have obtained formulas which express the classical characteristics of the surfaces S_τ: the principal directions, the principal curvatures, the mean curvature, and the Gaussian curvature in terms of the classical characteristics of the curves L_τ, i.e., their Frenet basis, the first curvature, and the second curvature. A new proof for the equality of the non-holonomicity values of the fields of principal directions has been obtained. The proofs are based on the fact that the principal curvatures are stationary values of the normal curvature at each point of the surface S_τ.

The vector lines L_τ of the physical vector fields corresponding to the solutions of the equations of mathematical physics (curves L_τ have the unit tangent vector $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$) form the family of curves $\{L_\tau\}$ and continuously fill in the domain considered. For example, for the solutions τ of the eikonal equation $\tau_x^2 + \tau_y^2 + \tau_z^2 = n^2(x, y, z)$ (here $\tau = \tau(x, y, z)$ is the scalar time field and n is the refractive index), which is the basic mathematical model of kinematic seismics (geometric optics), the role of the curves L_τ being played by the rays, i.e., the vector lines of the field $\boldsymbol{v} = \operatorname{grad} \tau = n\boldsymbol{\tau}$. For the Euler hydrodynamic equations, the role of the curves L_τ is played by streamlines.

In mathematical physics, there often occur situations where, along with the family of curves $\{L_\tau\}$, the family $\{S_\tau\}$ of surfaces S_τ with the unit normal $\boldsymbol{\tau}$ which are orthogonal to the curves L_τ (the field $\boldsymbol{\tau}$) exists and is studied. For example, for the eikonal equation, the role of the surfaces S_τ is played by the wavefronts $\tau(x, y, z) = \text{const}$ orthogonal to the family of rays $\{L_\tau\}$. Therefore, in this paper, we study the properties of the families of curves $\{L_\tau\}$ and surfaces $\{S_\tau\}$ which are mutually orthogonal and are considered simultaneously, rather than the properties of fixed curves and surfaces.

The basic characteristics of the curves L_τ of classical differential geometry [2–4] are the Frenet basis $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$, where $\boldsymbol{\tau}$ is the unit tangent vector, $\boldsymbol{\nu}$ is the principal normal, and $\boldsymbol{\beta}$ is the binormal, the first curvature k, and the second curvature $\varkappa$, which are defined at each point of a given curve. The most important classical characteristics of the surface are its unit normal $\boldsymbol{\tau}$, the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, the principal curvatures k_1 and k_2, the mean curvature $H \stackrel{\text{def}}{=} (k_1 + k_2)/2$, and the Gaussian curvature $K \stackrel{\text{def}}{=} k_1 k_2$, which are defined at each point of a given surface. For the families $\{L_\tau\}$ and $\{S_\tau\}$, all the quantities $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, k, $\varkappa$ and $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, H, and K are the vector and the scalar fields in the domain D continuously filled with the curves L_τ and the surfaces S_τ. The symbols $\boldsymbol{a}\cdot\boldsymbol{b}$ and $\boldsymbol{a}\times\boldsymbol{b}$ denote the scalar and vector products of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, ∇ is the Hamiltonian operator, $(\boldsymbol{v}\cdot\nabla)\boldsymbol{a}$ is the derivative of the vector $\boldsymbol{a}$ in the direction of the vector $\boldsymbol{v}$.

In this paper, we prove the formulas which express the characteristics $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, k_1, k_2, H, and K of the surfaces $S_\tau \in \{S_\tau\}$ in terms of the characteristics $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, and $\varkappa$ of the curves $L_\tau \in \{L_\tau\}$ orthogonal to the surfaces S_τ. In addition, a new proof for the following property of the family of surfaces stated in [1] is obtained: the non-holonomicity values of the fields of the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are equal. (The non-holonomicity value of the unit vector field $\boldsymbol{\tau}$ is the quantity $\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau}$. The condition $\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau} = 0$ is the necessary and sufficient condition for holonomicity of the field $\boldsymbol{\tau}$, i.e., for the existence of a family of surfaces orthogonal to the field $\boldsymbol{\tau}$, i.e., to its vector lines L_τ [5, Ch. 1, § 1].) In addition, another proof of the formula $K = \boldsymbol{\tau}\cdot(\operatorname{rot}\boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\beta}) - \varkappa^2$ derived in [1] is given.

Let us assume that $\{L_\tau\}$ is a family of curves L_τ which continuously fill in the domain D, and

(A) one and only one curve $L_\tau \in \{L_\tau\}$ passes through each point $(x, y, z) \in D$;

(B) at each point (x, y, z) of any curve $L_\tau \in \{L_\tau\}$, the right-hand Frenet basis $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$ exists, so that the three mutually orthogonal vector fields $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ are defined in D, and $\boldsymbol{\tau} = \boldsymbol{\nu}\times\boldsymbol{\beta}$, $\boldsymbol{\nu} = \boldsymbol{\beta}\times\boldsymbol{\tau}$, $\boldsymbol{\beta} = \boldsymbol{\tau}\times\boldsymbol{\nu}$;

(C) $\boldsymbol{\tau} \in C^2(D)$.

In D, let there exist a family of surfaces S_τ orthogonal to the family of curves $\{L_\tau\}$, i.e., to the field $\boldsymbol{\tau}$, which, according to the Jacobi theorem [1, Ch. 1, § 1], is equivalent to the identity $\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau} = 0$ in D. Therefore, $\{L_\tau\}$ is the family of vector lines of the field of normals $\boldsymbol{\tau}$ to the surfaces S_τ. Let $\{S_\tau\}$ be the family of surfaces S_τ with the unit normal $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$, which continuously fill in the domain D in the space x, y, z. The principal direction will be represented by the unit vector $\boldsymbol{l}_i$ $(i = 1, 2)$ with the corresponding direction; the vector $\boldsymbol{l}_i$ is the tangent unit vector of the curvature line L_i on S_τ, and at the point $(x, y, z) \in S_\tau$, it is equal to the derivative of the radius vector $\boldsymbol{r} = \boldsymbol{r}(x, y, z)$ of the point of the surface S_τ in the principal direction at the point (x, y, z). Let us assume that

(D) one and only one surface $S_\tau \in \{S_\tau\}$ passes through each point $(x, y, z) \in D$;

(E) at each point $(x, y, z) \in D$, there exists a right-hand system of mutually orthogonal unit vectors $\boldsymbol{\tau}$, $\boldsymbol{l}_1$, and $\boldsymbol{l}_2$, where $\boldsymbol{\tau}$ is the unit normal and $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are the principal directions at the surface S_τ passing through this point. For this, it is sufficient that each surface $S_\tau \in \{S_\tau\}$ be C^2-regular [3]. Thus, in D, we have defined three mutually orthogonal unit vector fields $\boldsymbol{\tau}(x, y, z)$, $\boldsymbol{l}_1(x, y, z)$, $\boldsymbol{l}_2(x, y, z)$; $\boldsymbol{l}_1 = \boldsymbol{l}_2 \times \boldsymbol{\tau}$, $\boldsymbol{l}_2 = \boldsymbol{\tau} \times \boldsymbol{l}_1$, $\boldsymbol{\tau} = \boldsymbol{l}_1 \times \boldsymbol{l}_2$;

(F) $\boldsymbol{\tau} \in C^2(D)$, $\boldsymbol{l}_1, \boldsymbol{l}_2 \in C^1(D)$.

Theorem 1. *Suppose that, for the family $\{L_\tau\}$ of curves L_τ with the unit tangent vector $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y, z)$, conditions* (A)–(C) *are satisfied in the domain D and that $\{S_\tau\}$ is the family of surfaces S_τ with unit normal $\boldsymbol{\tau}$ which are orthogonal to the family $\{L_\tau\}$. Let the family $\{S_\tau\}$ satisfy conditions* (D)–(F) *in the domain D. Then, at each point $(x, y, z) \in D$, the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ of the surface S_τ passing through this point are expressed in terms of the Frenet unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ of the curves L_τ by the formulas*

$$\boldsymbol{l}_1 = \boldsymbol{\nu} \cos \omega + \boldsymbol{\beta} \sin \omega, \quad \boldsymbol{l}_2 = -\boldsymbol{\nu} \sin \omega + \boldsymbol{\beta} \cos \omega, \tag{1}$$

where $\omega = \omega(x, y, z)$ is a scalar function (ω is the angle between the vectors $\boldsymbol{l}_1$ and $\boldsymbol{\nu}$ or between $\boldsymbol{l}_2$ and $\boldsymbol{\beta}$). In addition, the fields of the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ in the domain D satisfy the identity

$$\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_1 = \boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_2. \tag{2}$$

In terms of the geometry of vector fields [5, Ch. 1, § 1]*, identity* (2) *implies that the non-holonomicity values of the vector fields of the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are equal in D. Identity* (2) *is equivalent to the condition*

$$\operatorname{tg} 2\omega = -\frac{A}{B}, \tag{3}$$

in D, which defines the function ω in terms of $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$. Here $A \overset{\text{def}}{=} \boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\nu} - \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\beta}$, $B \overset{\text{def}}{=} \boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\nu} + \boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta}$. For the principal curvatures k_1 and k_2 of the surfaces S_τ, the following formulas are valid:

$$k_1 = -\operatorname{rot} \boldsymbol{l}_1 \cdot \boldsymbol{l}_2, \quad k_2 = \operatorname{rot} \boldsymbol{l}_2 \cdot \boldsymbol{l}_1. \tag{4}$$

Proof. Let $M(x, y, z)$ be an arbitrary point of the domain D, and let L_τ and S_τ be the curve of the family $\{L_\tau\}$ and the surface of the family $\{S_\tau\}$, respectively, that pass through this point. The principal normal $\boldsymbol{\nu}$ and the binormal $\boldsymbol{\beta}$ of the curve L_τ are in a plane normal to the curve L_τ passing through the point M, and the principal directions $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are in a plane tangent to the surface S_τ. Because the families $\{L_\tau\}$ and $\{S_\tau\}$ are mutually orthogonal, these planes coincide and the unit vectors $\boldsymbol{\nu}$, $\boldsymbol{\beta}$, $\boldsymbol{l}_1$, and $\boldsymbol{l}_2$ are in the same plane. In addition, the vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are mutually orthogonal. Therefore, at each point $M \in D$, the vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ can be represented in the form of (1). Because $\boldsymbol{l}_i$ is the unit tangent vector of the curvature line L_i at the surface S passing through the point M, it follows that the curvature vector K_i of the curve L_i equals $K_i = \operatorname{rot}\boldsymbol{l}_i \times \boldsymbol{l}_i$ and the normal (principal) curvature of the curve L_i is $k_i = \boldsymbol{\tau} \cdot \boldsymbol{K}_i = \boldsymbol{\tau} \cdot (\operatorname{rot}\boldsymbol{l}_i \times \boldsymbol{l}_i)$. This implies formulas (4): $k_1 = \boldsymbol{\tau} \cdot (\operatorname{rot}\boldsymbol{l}_1 \times \boldsymbol{l}_1) = -\operatorname{rot}\boldsymbol{l}_1 \cdot (\boldsymbol{\tau} \times \boldsymbol{l}_1) = -\boldsymbol{l}_2 \cdot \operatorname{rot}\boldsymbol{l}_1$, $k_2 = \boldsymbol{\tau} \cdot (\operatorname{rot}\boldsymbol{l}_2 \times \boldsymbol{l}_2) = -\operatorname{rot}\boldsymbol{l}_2 \cdot (\boldsymbol{\tau} \times \boldsymbol{l}_2) = \boldsymbol{l}_1 \cdot \operatorname{rot}\boldsymbol{l}_2$. We make use of the well-known formula $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = -\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{c})$ and the equalities $\boldsymbol{\tau} \times \boldsymbol{l}_1 = \boldsymbol{l}_2$, $\boldsymbol{\tau} \times \boldsymbol{l}_2 = -\boldsymbol{l}_1$ [2].

The principal curvatures k_1 and k_2 are the stationary values of the normal curvature at the point of the surface [3, Ch. 2, § 4; 4, Ch. 2, § 3]. At a surface S_τ we will consider the curves $L_{1\varepsilon}$ and $L_{2\varepsilon}$ which pass through the point M, are mutually orthogonal at this point, and are close to the curvature lines L_1 and L_2, respectively. We denote the normal curvature of the curve $L_{i\varepsilon}$ by $k_{i\varepsilon}$, and its unit tangent vector and the curvature vector by $\boldsymbol{l}_{i\varepsilon}$ and by $\boldsymbol{K}_{i\varepsilon}$, respectively. The formulas for $\boldsymbol{l}_{i\varepsilon}$ are obtained from formula (1) replacing $\boldsymbol{l}_i$ by $\boldsymbol{l}_{i\varepsilon}$ and replacing ω by $\tilde{\omega} = \omega + \varepsilon\eta$, where ε is a small parameter and η is a fixed arbitrary smooth function. We seek stationary values of the normal curvatures $k_{1\varepsilon}$ and $k_{2\varepsilon}$ of the curves $L_{1\varepsilon}$ and $L_{2\varepsilon}$ by varying $\tilde{\omega}$ due to a variation in the parameter ε in the neighborhood of the point $\varepsilon = 0$. We have $\dfrac{\partial \boldsymbol{l}_{1\varepsilon}}{\partial\varepsilon} = \eta\boldsymbol{l}_{2\varepsilon}$, $\dfrac{\partial \boldsymbol{l}_{1\varepsilon}}{\partial\varepsilon}\Big|_{\varepsilon=0} = \eta\boldsymbol{l}_2$, $\dfrac{\partial \boldsymbol{l}_{2\varepsilon}}{\partial\varepsilon} = -\eta\boldsymbol{l}_{1\varepsilon}$, $\dfrac{\partial \boldsymbol{l}_{2\varepsilon}}{\partial\varepsilon}\Big|_{\varepsilon=0} = -\eta\boldsymbol{l}_1$, $\dfrac{\partial \operatorname{rot}\boldsymbol{l}_{1\varepsilon}}{\partial\varepsilon} = \operatorname{rot}\dfrac{\partial \boldsymbol{l}_{1\varepsilon}}{\partial\varepsilon} = \operatorname{rot}(\eta\boldsymbol{l}_{2\varepsilon})$, $\dfrac{\partial \operatorname{rot}\boldsymbol{l}_{1\varepsilon}}{\partial\varepsilon}\Big|_{\varepsilon=0} = \operatorname{rot}(\eta\boldsymbol{l}_2) = \eta\operatorname{rot}\boldsymbol{l}_2 + \operatorname{grad}\eta \times \boldsymbol{l}_2$, $\dfrac{\partial \operatorname{rot}\boldsymbol{l}_{2\varepsilon}}{\partial\varepsilon} = \operatorname{rot}\dfrac{\partial \boldsymbol{l}_{2\varepsilon}}{\partial\varepsilon} = -\operatorname{rot}(\eta\boldsymbol{l}_{1\varepsilon})$, $\dfrac{\partial \operatorname{rot}\boldsymbol{l}_{2\varepsilon}}{\partial\varepsilon}\Big|_{\varepsilon=0} = -\operatorname{rot}(\eta\boldsymbol{l}_1) = -\eta\operatorname{rot}\boldsymbol{l}_1 - \operatorname{grad}\eta \times \boldsymbol{l}_1$. Hence, the normal curvature $k_{1\varepsilon}$ of the curves $L_{1\varepsilon}$ is expressed as $k_{1\varepsilon} = \boldsymbol{\tau} \cdot \boldsymbol{K}_{1\varepsilon} = \boldsymbol{\tau} \cdot (\operatorname{rot}\boldsymbol{l}_{1\varepsilon} \times \boldsymbol{l}_{1\varepsilon})$, $\dfrac{\partial k_{1\varepsilon}}{\partial\varepsilon} = \dfrac{\partial}{\partial\varepsilon}[\boldsymbol{\tau} \cdot (\operatorname{rot}\boldsymbol{l}_{1\varepsilon} \times \boldsymbol{l}_{1\varepsilon})] = \boldsymbol{\tau} \cdot \dfrac{\partial}{\partial\varepsilon}(\operatorname{rot}\boldsymbol{l}_{1\varepsilon} \times \boldsymbol{l}_{1\varepsilon}) = \boldsymbol{\tau} \cdot \left(\dfrac{\partial}{\partial\varepsilon}\operatorname{rot}\boldsymbol{l}_{1\varepsilon} \times \boldsymbol{l}_{1\varepsilon} + \operatorname{rot}\boldsymbol{l}_{1\varepsilon} \times \dfrac{\partial \boldsymbol{l}_{1\varepsilon}}{\partial\varepsilon}\right) \Rightarrow \dfrac{\partial k_{1\varepsilon}}{\partial\varepsilon}\Big|_{\varepsilon=0} = \boldsymbol{\tau} \cdot [(\eta\operatorname{rot}\boldsymbol{l}_2 + \operatorname{grad}\eta \times \boldsymbol{l}_2) \times \boldsymbol{l}_1 + \operatorname{rot}\boldsymbol{l}_1 \times \eta\boldsymbol{l}_2] = \eta[\boldsymbol{\tau} \cdot (\operatorname{rot}\boldsymbol{l}_1 \times \boldsymbol{l}_2 + \operatorname{rot}\boldsymbol{l}_2 \times \boldsymbol{l}_1)]$. Here we used the well-known formula $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{b} \cdot \boldsymbol{a})$ [2], which implies that $(\operatorname{grad}\eta \times \boldsymbol{l}_2) \times \boldsymbol{l}_1 = -\boldsymbol{l}_1 \times (\operatorname{grad}\eta \times \boldsymbol{l}_2) = \boldsymbol{l}_2(\operatorname{grad}\eta \cdot \boldsymbol{l}_1) - \operatorname{grad}\eta(\boldsymbol{l}_1 \cdot \boldsymbol{l}_2)$, and the equalities $\boldsymbol{l}_1 \cdot \boldsymbol{l}_2 = 0$ and $\boldsymbol{\tau} \cdot \boldsymbol{l}_2 = 0$. Using the same well-known formula and the equalities $\boldsymbol{l}_2 = \boldsymbol{\tau} \times \boldsymbol{l}_1$ and $\boldsymbol{l}_1 = \boldsymbol{l}_2 \times \boldsymbol{\tau}$, we obtain $\operatorname{rot}\boldsymbol{l}_1 \times \boldsymbol{l}_2 + \operatorname{rot}\boldsymbol{l}_2 \times \boldsymbol{l}_1 = \boldsymbol{\tau}(\operatorname{rot}\boldsymbol{l}_1 \cdot \boldsymbol{l}_1) - \boldsymbol{l}_1(\operatorname{rot}\boldsymbol{l}_1 \cdot \boldsymbol{\tau}) + \boldsymbol{l}_2(\operatorname{rot}\boldsymbol{l}_2 \cdot \boldsymbol{\tau}) - \boldsymbol{\tau}(\operatorname{rot}\boldsymbol{l}_2 \cdot \boldsymbol{l}_2)$. It follows that $\dfrac{\partial k_{1\varepsilon}}{\partial\varepsilon}\Big|_{\varepsilon=0} = \eta(\boldsymbol{l}_1 \cdot \operatorname{rot}\boldsymbol{l}_1 - \boldsymbol{l}_2 \cdot \operatorname{rot}\boldsymbol{l}_2)$. Consequently, the stationarity condition $\dfrac{\partial k_{1\varepsilon}}{\partial\varepsilon}\Big|_{\varepsilon=0} = 0$ is equivalent to the equality $\boldsymbol{l}_1 \cdot \operatorname{rot}\boldsymbol{l}_1 - \boldsymbol{l}_2 \cdot \operatorname{rot}\boldsymbol{l}_2 = 0$ in D, which leads to identity (2) of the theorem.

The stationarity condition $\left.\frac{\partial k_{2\varepsilon}}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$ results in the same identity (2).

Let us show that identity (2) is equivalent to identity (3). Using the well-known formula [2] $\operatorname{rot}(\varphi \boldsymbol{a}) = \varphi \operatorname{rot} \boldsymbol{a} + \operatorname{grad} \varphi \times \boldsymbol{a}$, from (1) it follows that $\operatorname{rot} \boldsymbol{l}_1 = \cos\omega \operatorname{rot} \boldsymbol{\nu} + \sin\omega \operatorname{rot} \boldsymbol{\beta} + \operatorname{grad} \omega \times \boldsymbol{l}_2$ and $\operatorname{rot} \boldsymbol{l}_2 = -\sin\omega \operatorname{rot} \boldsymbol{\nu} + \cos\omega \operatorname{rot} \boldsymbol{\beta} - \operatorname{grad} \omega \times \boldsymbol{l}_1$. From this, after lengthy but simple calculations, we obtain $\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_1 - \boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_2 = A\cos 2\omega + B\sin 2\omega$ and, in view of identity (2), we arrive at formula (3). □

Corollary 1. *Let the family* $\{S_\tau\}$ *of the surfaces* S_τ *satisfy conditions* (D)–(F) *in the domain* D. *Then, the Gaussian curvature* K *of the surfaces* S_τ *is expressed in terms of the principal directions* $\boldsymbol{l}_1$ *and* $\boldsymbol{l}_2$ *by the formulas*

$$K \overset{\text{def}}{=} k_1 k_2 = -(\operatorname{rot} \boldsymbol{l}_1 \cdot \boldsymbol{l}_2)(\operatorname{rot} \boldsymbol{l}_2 \cdot \boldsymbol{l}_1) = \boldsymbol{\tau} \cdot (\operatorname{rot} \boldsymbol{l}_1 \times \operatorname{rot} \boldsymbol{l}_2) - (\operatorname{rot} \boldsymbol{l}_i \cdot \boldsymbol{l}_i)^2, \quad (5)$$

where $i = 1$ *or* 2 *and for the mean curvature* H, *the formula* $H = -\operatorname{div} \boldsymbol{\tau}/2$ *from* [5, § 5] *holds.*

Proof. From (4) we obtain $K = k_1 k_2 = -(\operatorname{rot} \boldsymbol{l}_1 \cdot \boldsymbol{l}_2)(\operatorname{rot} \boldsymbol{l}_2 \cdot \boldsymbol{l}_1) = (\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_1)(\boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_2) - (\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_2)(\boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_1) - (\boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_1)(\boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_2)$. Using the well-known formula $(\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \cdot \boldsymbol{d}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d})$ for $\boldsymbol{a} = \boldsymbol{l}_1$, $\boldsymbol{b} = \boldsymbol{l}_2$, $\boldsymbol{c} = \operatorname{rot} \boldsymbol{l}_1$, and $\boldsymbol{d} = \operatorname{rot} \boldsymbol{l}_2$ [2, § 7] and equality (2), we obtain formula (5). From (4) it follows that $k_1 + k_2 \overset{\text{def}}{=} 2H = -\operatorname{rot} \boldsymbol{l}_1 \cdot \boldsymbol{l}_2 + \operatorname{rot} \boldsymbol{l}_2 \cdot \boldsymbol{l}_1 = -\operatorname{div}(\boldsymbol{l}_1 \times \boldsymbol{l}_2) = -\operatorname{div} \boldsymbol{\tau}$, which should be proved. Here we used the well-known formula $\operatorname{div}(\boldsymbol{a} \times \boldsymbol{b}) = \operatorname{rot} \boldsymbol{a} \cdot \boldsymbol{b} - \operatorname{rot} \boldsymbol{b} \cdot \boldsymbol{a}$ for $\boldsymbol{a} = \boldsymbol{l}_1$, $\boldsymbol{b} = \boldsymbol{l}_2$ [2, § 17]. □

Theorem 2. *Let* $\{L_\tau\}$ *and* $\{S_\tau\}$ *be mutually orthogonal families of the curves* L_τ *and the surfaces* S_τ *in the domain* D, *so that the field of the unit tangent vectors* $\boldsymbol{\tau}$ *of the curves* L_τ *and the field of the normals* $\boldsymbol{\tau}$ *of the surfaces* S_τ *coincide. Let for the families* $\{L_\tau\}$ *and* $\{S_\tau\}$, *conditions* (A)–(C) *and* (D)–(F), *respectively, be satisfied. Then at each point* $(x, y, z) \in D$, *the principal curvatures* k_1 *and* k_2, *the mean curvature* H, *and the Gaussian curvature* K *of the surface* S_τ *passing through this point are expressed in terms of the Frenet unit vectors* $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, *and* $\boldsymbol{\beta}$ *of the curves* L_τ *by the formulas*

$$\begin{aligned} k_1 &= \frac{1}{2}\big(-\operatorname{div} \boldsymbol{\tau} \pm \sqrt{A^2 + B^2}\big) = -\boldsymbol{l}_2 \cdot \operatorname{rot} \boldsymbol{l}_1, \\ k_2 &= \frac{1}{2}\big(-\operatorname{div} \boldsymbol{\tau} \mp \sqrt{A^2 + B^2}\big) = \boldsymbol{l}_1 \cdot \operatorname{rot} \boldsymbol{l}_2, \end{aligned} \quad (6)$$

$$\Rightarrow \quad K \overset{\text{def}}{=} k_1 k_2 = \frac{1}{4}\big[(\operatorname{div} \boldsymbol{\tau})^2 - (A^2 + B^2)\big], \quad (7)$$

$$H \overset{\text{def}}{=} \frac{k_1 + k_2}{2} = -\frac{1}{2}(\operatorname{rot} \boldsymbol{\nu} \cdot \boldsymbol{\beta} - \operatorname{rot} \boldsymbol{\beta} \cdot \boldsymbol{\nu}) = -\frac{1}{2} \operatorname{div} \boldsymbol{\tau}, \quad (8)$$

where the quantities A and B are defined by the formulas from Theorem 1. The upper sign in front of the radical is taken for $k_1 > k_2$ and the lower sign — for $k_1 < k_2$.

Proof. Substitution of the expressions for $\operatorname{rot}\boldsymbol{l}_1$ and $\operatorname{rot}\boldsymbol{l}_2$ contained in the proof of Theorem 1 into equalities (4) yields $k_1 = -\operatorname{rot}\boldsymbol{l}_1\cdot\boldsymbol{l}_2 = A\sin\omega\cos\omega - (\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta})\cos^2\omega + (\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu})\sin^2\omega$, $k_2 = \operatorname{rot}\boldsymbol{l}_2\cdot\boldsymbol{l}_1 = -A\sin\omega\cos\omega - (\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta})\sin^2\omega + (\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu})\cos^2\omega = -k_1 - (\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta} - \operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu}) = -k_1 - \operatorname{div}\boldsymbol{\tau}$. Here we used the well-known formula $\operatorname{div}(\boldsymbol{a}\times\boldsymbol{b}) = \operatorname{rot}\boldsymbol{a}\cdot\boldsymbol{b} - \operatorname{rot}\boldsymbol{b}\cdot\boldsymbol{a}$ [2] for the vector $\boldsymbol{\tau} = \boldsymbol{\nu}\times\boldsymbol{\beta}$. Combining and subtracting the equalities for k_1 and k_2, we obtain formula (8): $k_1 + k_2 \overset{\text{def}}{=} 2H = -\operatorname{div}\boldsymbol{\tau} = \operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta} - \operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu}$ and the equality $k_1 - k_2 = A\sin 2\omega - B\cos 2\omega$. Again, we have obtained the proof of the formula $H = -\operatorname{div}\boldsymbol{\tau}/2$ from [5, § 5].

Combining the latter formula and equality (3) brings about $A\sin 2\omega - B\cos 2\omega = k_1 - k_2$ and $A\cos 2\omega + B\sin 2\omega = 0$. Taking the square of these two equalities and combining the results, we obtain $(k_1 - k_2)^2 = A^2 + B^2 \Rightarrow k_1 - k_2 = \pm\sqrt{A^2 + B^2}$, where the upper sign in front of the radical is taken for $k_1 > k_2$ and the lower sign — for $k_1 < k_2$. Combining and subtracting the latter equality and the formula $k_1 + k_2 = -\operatorname{div}\boldsymbol{\tau}$, we obtain equalities (6), which immediately resulted in expression (7) for the Gaussian curvature K. □

Lemma. *Let the family of curves $\{L_\tau\}$ with the Frenet unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ and second curvature $\varkappa$ satisfy conditions* (A)–(C) *in a domain D. Then, in D, we have the identity $A^2 + B^2 = (\operatorname{div}\boldsymbol{\tau})^2 + 4\Big[\Big(\varkappa - \frac{1}{2}\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau}\Big)^2 - \boldsymbol{\tau}\cdot(\operatorname{rot}\boldsymbol{\nu}\times\operatorname{rot}\boldsymbol{\beta})\Big]$, where the quantities A and B are defined by the formulas from Theorem 1.*

Proof. From the definition A and the formula $\varkappa = (\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau} - \boldsymbol{\nu}\cdot\operatorname{rot}\boldsymbol{\nu} - \boldsymbol{\beta}\cdot\operatorname{rot}\boldsymbol{\beta})/2$ [5, Ch. 1, § 15], we obtain $A^2 = (\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\nu})^2 - 2(\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\nu})(\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\beta}) + (\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\beta})^2$ and $(2\varkappa - \boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau})^2 = (\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\nu})^2 + 2(\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\nu})(\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\beta}) + (\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\beta})^2$, whence $A^2 = (2\varkappa - \boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau})^2 - 4(\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\nu})(\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\beta})$. From the definition of B and the formula $\operatorname{div}\boldsymbol{\tau} = \operatorname{div}(\boldsymbol{\nu}\times\boldsymbol{\beta}) = \operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta} - \operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu}$, we obtain $B^2 = (\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta})^2 + 2(\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta})(\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu}) + (\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu})^2$ and $(\operatorname{div}\boldsymbol{\tau})^2 = (\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta})^2 - 2(\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta})(\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu}) + (\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu})^2$, whence $B^2 = (\operatorname{div}\boldsymbol{\tau})^2 + 4(\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta})(\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu})$. Combining the expressions obtained for A^2 and B^2 leads to $A^2 + B^2 = (\operatorname{div}\boldsymbol{\tau})^2 + 4\Big(\varkappa - \frac{1}{2}\boldsymbol{\tau}\cdot\operatorname{rot}\boldsymbol{\tau}\Big)^2 + 4\big[(\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\beta})(\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\nu}) - (\operatorname{rot}\boldsymbol{\nu}\cdot\boldsymbol{\nu})(\operatorname{rot}\boldsymbol{\beta}\cdot\boldsymbol{\beta})\big]$. Using the well-known formula $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$ [2, § 7] for $\boldsymbol{a} = \boldsymbol{\nu}$, $\boldsymbol{b} = \boldsymbol{\beta}$, $\boldsymbol{c} = \operatorname{rot}\boldsymbol{\nu}$, and $\boldsymbol{d} = \operatorname{rot}\boldsymbol{\beta}$ and the equality $\boldsymbol{\tau} = \boldsymbol{\nu}\times\boldsymbol{\beta}$, we obtain the lemma. □

Theorem 3. *Let $\{L_\tau\}$ and $\{S_\tau\}$ be the mutually orthogonal families of curves L_τ and surfaces S_τ in a domain D and let the conditions of Theorem 2 be satisfied. Then, at each point $M(x,y,z) \in D$, the Gaussian curvature of the surface S_τ passing through this point is expressed in terms of the Frenet unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ and the second curvature $\varkappa$ of the curves L_τ — by any of the formulas*

$$K = \boldsymbol{\tau} \cdot (\operatorname{rot} \boldsymbol{\nu} \times \operatorname{rot} \boldsymbol{\beta}) - \varkappa^2 \tag{9}$$

$$\Leftrightarrow \quad K = -\Big[(\boldsymbol{\nu} \cdot \operatorname{rot} \boldsymbol{\beta})(\boldsymbol{\beta} \cdot \operatorname{rot} \boldsymbol{\nu}) + \frac{1}{4}A^2\Big]. \tag{10}$$

Proof. Since the field $\boldsymbol{\tau}$ is holonomic, i.e., there exists a family of surfaces S_τ orthogonal to the field $\boldsymbol{\tau}$, it follows from the Jacobi theorem that the identity $\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$ holds in the domain D. Using this identity and substituting the formula for $A^2 + B^2$ from the lemma into equality (7), we come to (9). From the latter, using the expressions for A^2 in terms of $\varkappa$, $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, and $\boldsymbol{\beta}$ contained in the proof of the lemma, we obtain (10). In [1, Sec. 3.2], formula (9) was derived using a different proof. □

Remark 1. In [1, Sec. 2.3], the following formula was derived for the unit vector field $\boldsymbol{\tau}$: $\frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = \varkappa(\varkappa - \boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau}) - \boldsymbol{\tau} \cdot (\operatorname{rot} \boldsymbol{\nu} \times \operatorname{rot} \boldsymbol{\beta})$, where $\boldsymbol{S}(\boldsymbol{\tau}) = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau} - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} = \boldsymbol{K}_\tau + 2H\boldsymbol{\tau}$, $\boldsymbol{K}_\tau = k\boldsymbol{\nu} = \operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau}$ is the curvature vector of the vector lines L_τ of the field $\boldsymbol{\tau}$ and H is the mean curvature. Comparing this formula for the case $\boldsymbol{\tau} \cdot \operatorname{rot} \boldsymbol{\tau} = 0$ (i.e., for the holonomic field $\boldsymbol{\tau}$) with formula (9), we obtain $K = -\frac{1}{2} \operatorname{div} \boldsymbol{S}(\boldsymbol{\tau})$, i.e., the second divergent representation of the Gaussian curvature [5, Ch. 1, § 8] using the new proof.

Remark 2. The Frenet unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ and the curvature k of the curves L_τ can be expressed in terms of $\boldsymbol{\tau}$: $\boldsymbol{\nu} = (\operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau})/k$, $\boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$, and $k = |\operatorname{rot} \boldsymbol{\tau} \times \boldsymbol{\tau}|$, respectively. Therefore, all formulas for the quantities $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, ω, k_1, k_2, and K in Theorems 1–3 and the lemma can be expressed in terms of only the field $\boldsymbol{\tau}$ (the unit tangent vectors of the curves L_τ or normals to S_τ).

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