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Modeling the infrasonic and seismic waves propagation from various types of singular sources in the Atmosphere–Earth interface*

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Abstract. In this paper, we consider the results of the numerical simulation of the propagation of seismic and acousto-gravitational waves for the spatially inhomogeneous "Atmosphere–Earth" model. The seismic wave propagation in an elastic half-space is described by a system of first order dynamic equations of the elastic theory through interconnection of a component of displacement velocity vector and a component of the strain tensor. Acoustic-gravity waves propagation in the non-ionized isothermal atmosphere is described by the linearized Navier–Stokes equations. It is assumed that the wind is directed along the horizontal axis, and that the speed and direction of the wind depends on the height. For the numerical solution of the problem, the numerical method based on combining integral of the Laguerre and the Fourier transforms with a finite difference algorithm is used.

Keywords: seismic waves, acoustic-gravitational waves, Navier–Stokes equations, Laguerre transform, finite difference method.

Introduction

Currently, there are many scientific publications that show a high degree of the relationship between waves in the lithosphere and atmosphere. The published work [1] describes the effect of acousto-seismic induction, in which an acoustic wave from a vibrator, due to the phenomenon of refraction in the atmosphere, excites intense surface seismic waves at a distance of tens of kilometers. In turn, the lithospheric seismic waves from earthquakes and explosions generate atmospheric acoustic-gravity waves, which are especially intense in the upper layers of the atmosphere with a low density and in the ionosphere.

This paper considers an algorithm for solving and the results of numerical modeling of the problem of propagation and mutual generation of seismic and acoustic-gravity waves for a combined spatially inhomogeneous "Atmosphere–Earth" model. These studies are a continuation of the studies given in [2, 3]. In this paper, in contrast to these previous ones, an algorithm using the finite-difference approximation of the derivatives in two spatial coordinates is proposed, in which the simulated medium is assumed

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to be inhomogeneous, and in the third homogeneous coordinate, as before, the Fourier transform is used. This problem statement is usually called the 2.5D problem. The use of the Laguerre transform in the time coordinate for the numerical implementation of the solution of the problem can be considered as an analogue of the well-known spectral method based on the Fourier transform, where instead of the frequency ω we have the parameter p— the degree of the Laguerre polynomials. However, in contrast to the Fourier transform, the use of the integral Laguerre transform with respect to time makes it possible to reduce the original problem to solving a system of equations in which the separation parameter is present only on the right-hand side of the equations and has a recurrent dependence. This method for solving dynamic problems of the theory of elasticity was first considered in [4, 5], and then developed for viscoelasticity problems [6, 7]. In these publications, the distinctive features of this method from the accepted approaches are considered and the advantages of using the integral Laguerre transform in contrast to the difference method and the Fourier transform with respect to time are discussed.

1. Formulation of the problem

The propagation of acousto-gravitational waves in an isothermal atmosphere is described by a linearized system of the Navier–Stokes equations in the form of a first-order hyperbolic system for a three-dimensional Cartesian coordinate system in the form

$$\frac{\partial u_x}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}, \quad \frac{\partial u_y}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}, \quad \frac{\partial u_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} - \frac{\rho g}{\rho_0}, \tag{1}$$

$$\frac{\partial P}{\partial t} = c_0^2\left(\frac{\partial \rho}{\partial t} + u_z\frac{\partial \rho_0}{\partial z}\right) - u_z\frac{\partial P_0}{\partial z}, \tag{2}$$

$$\frac{\partial \rho}{\partial t} = -\rho_0\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - u_z\frac{\partial \rho_0}{\partial z}. \tag{3}$$

Here g is the acceleration of the force of gravity, $\rho_0(z)$ is the density of the unperturbed atmosphere, $c_0(z)$ is the speed of sound, $\vec{u} = (u_x, u_y, u_z)$ is the velocity vector of the displacement of air particles, P and ρ is, respectively, the perturbations of pressure and density under the action of wave propagation. Zero subindices for the physical parameters of the medium mean that their values are set for the undisturbed state of the atmosphere. The dependence of the atmospheric pressure P_0 and ρ_0 density for an undisturbed state of the atmosphere in a uniform gravitational field can be defined as

$$\frac{\partial P_0}{\partial z} = -\rho_0 g, \quad \rho_0(z) = \rho_1 \exp(-z/H),$$

where H is the height of the isothermal homogeneous atmosphere, and ρ_1 is the density of the atmosphere near the Earth's surface, i.e., at $z = 0$.

The propagation of seismic waves in an elastic medium is written down by the well-known system of equations of the first order of the theory of elasticity through the relationship between the components of the displacement velocity vector and the components of the stress tensor

$$\frac{\partial u_i}{\partial t} = \frac{1}{\rho_0}\frac{\partial \sigma_{ik}}{\partial x_k} + F_i f(t), \tag{4}$$

$$\frac{\partial \sigma_{ik}}{\partial t} = \mu\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right) + \lambda\delta_{ik}\operatorname{div}\vec{u}. \tag{5}$$

Here δ_{ik} is the Kronecker symbol, $\lambda(x_1, x_2, x_3)$ and $\mu(x_1, x_2, x_3)$ are the elastic parameters of the medium, $\rho_0(x_1, x_2, x_3)$ is the density of the medium, $\vec{u} = (u_1, u_2, u_3)$ is the displacement velocity vector, and σ_{ij} are the stress tensor components. The components F_i of the force vector $\vec{F}(x, y, z) = F_1\vec{e}_x + F_2\vec{e}_y + F_3\vec{e}_z$ describe the action of a source localized in space, and $f(t)$ is a given time signal in the source.

The values of the components F_i depend on the type of the simulated source:

1. For a source of the "vertical force" type

$$F_1 = F_2 = 0, \quad F_3 = \delta(x_1 - x_0)\delta(x_2 - y_0)\delta(x_3 - z_0);$$

2. For a source of the "center of pressure" type

$$F_1 = \frac{\partial\delta(x_1 - x_0)}{\partial x_1}\delta(x_2 - y_0)\delta(x_3 - z_0),$$
$$F_2 = \delta(x_1 - x_0)\frac{\partial\delta(x_2 - y_0)}{\partial x_2}\delta(x_3 - z_0),$$
$$F_3 = \delta(x_1 - x_0)\delta(x_2 - y_0)\frac{\partial\delta(x_3 - z_0)}{\partial x_3};$$

3. For a source of the "dipole without moment" type

$$F_1 = F_2 = 0, \quad F_3 = \delta(x_1 - x_0)\delta(x_2 - y_0)\frac{\partial\delta(x_3 - z_0)}{\partial x_3}.$$

Here x_0, y_0, z_0 are the spatial coordinates of the source.

We assume that the interface between the atmosphere and the elastic half-space passes along the plane $z = 0$. In this case, the condition of contact of two media at $z = 0$ is written down as

$$u_z|_{z=-0} = u_z|_{z=+0};$$
$$\left.\frac{\partial \sigma_{zz}}{\partial t}\right|_{z=-0} = \left.\left(\frac{\partial \sigma_{zz}}{\partial t} + \rho_0 g u_z\right)\right|_{z=+0}; \quad \sigma_{xz}|_{z=-0} = \sigma_{yz}|_{z=-0} = 0. \tag{6}$$

The problem is solved with zero initial data

$$u_i|_{t=0} = \sigma_{ij}|_{t=0} = P|_{t=0} = \rho|_{t=0} = 0, \quad i = 1,2,3, \quad j = 1,2,3. \tag{7}$$

2. Solution algorithm

At the first stage of the solution, following [2, 3], we use the finite cosine-sine Fourier transform in the spatial coordinate, in whose direction the medium is considered to be homogeneous. For each component of the system, we introduce the corresponding cosine or sine transformation:

$$\overrightarrow{W}(x,z,n,t) = \int_0^a \overrightarrow{W}(x,y,z,t) \begin{Bmatrix} \cos(k_n y) \\ \sin(k_n y) \end{Bmatrix} dy, \quad n = 0,1,2,\ldots,N,$$

with the corresponding conversion formula

$$\overrightarrow{W}(x,y,z,t) = \frac{1}{\pi}\overrightarrow{W}(x,0,z,t) + \frac{2}{\pi}\sum_{n=1}^{N} \overrightarrow{W}(x,n,z,t)\cos(k_n y), \tag{8}$$

or

$$\overrightarrow{W}(x,y,z,t) = \frac{2}{\pi}\sum_{n=1}^{N} \overrightarrow{W}(x,n,z,t)\sin(k_n y), \tag{9}$$

where $k_n = \dfrac{n\pi}{a}$.

Let us choose the distance a to be large enough and consider the wave field up to the time instant $t < T$, where T is the minimum propagation time of the longitudinal wave to the boundary $r = a$. As a result of this transformation, we obtain $N+1$ independent of two-dimensional non-stationary problems with respect to space.

At the second stage of the solution, to the independent $N+1$ problems obtained in this way, we apply the integral Laguerre transform with respect to time of the form:

$$\overrightarrow{W}_p(x,n,z) = \int_0^\infty \overrightarrow{W}(x,n,z,t)(ht)^{-\alpha/2} l_p^\alpha(ht)\,d(ht), \quad p = 0,1,2,\ldots$$

with the inversion formula

$$\overrightarrow{W}(x,n,z,t) = (ht)^{\alpha/2}\sum_{p=0}^{\infty} \frac{p!}{(p+\alpha)!}\overrightarrow{W}_p(x,n,z) l_p^\alpha(ht), \tag{10}$$

where $l_p^\alpha(ht)$ are the orthogonal Laguerre functions.

The Laguerre functions $l_p^\alpha(ht)$ are expressed in terms of the classical orthonormal Laguerre polynomials $L_p^\alpha(ht)$ [8]. Here we choose α (the order of the Laguerre functions) integer and positive, then the following representation takes place:

$$l_p^\alpha(ht) = (ht)^{\alpha/2} e^{-ht/2} L_p^\alpha(ht).$$

To satisfy the initial conditions (7), it is necessary and sufficient to set $\alpha \geq 1$. In addition, a shift parameter $h > 0$ has been introduced, whose meaning and efficiency is discussed in detail in [5, 6].

As a result of these transformations, the solution to the original problem (4)–(7) is reduced to the solution of $N + 1$ independent of the two-dimensional differential problems in the spectral domain of the form:

$$\frac{h}{2}u_x^p - \frac{1}{\rho_0}\left(\frac{\partial \sigma_{xz}^p}{\partial z} + \frac{\partial \sigma_{xx}^p}{\partial x} + k_n \sigma_{xy}^p\right) = F_x(n) f^p - h\sum_{j=0}^{p-1} u_x^j, \tag{11}$$

$$\frac{h}{2}u_y^p - \frac{1}{\rho_0}\left(\frac{\partial \sigma_{yz}^p}{\partial z} + \frac{\partial \sigma_{xy}^p}{\partial x} - k_n \sigma_{yy}^p\right) = F_y(n) f^p - h\sum_{j=0}^{p-1} u_C^j, \tag{12}$$

$$\frac{h}{2}u_z^p - \frac{1}{\rho_0}\left(\frac{\partial \sigma_{zz}^p}{\partial z} + \frac{\partial \sigma_{xz}^p}{\partial x} + k_n \sigma_{yz}^p\right) + K_{\text{atm}}\frac{g}{\rho_0}\rho^p = F_z(n) f^p - h\sum_{j=0}^{p-1} u_z^j, \tag{13}$$

$$\frac{h}{2}\sigma_{xx}^p - \lambda\left(\frac{\partial u_z^p}{\partial z} + k_n u_y^p\right) - (\lambda + 2\mu)\frac{\partial u_x^p}{\partial x} + K_{\text{atm}}\rho_0 g u_z^p = -h\sum_{j=0}^{p-1} \sigma_{xx}^j, \tag{14}$$

$$\frac{h}{2}\sigma_{yy}^p - \lambda\left(\frac{\partial u_z^p}{\partial z} + \frac{\partial u_x^p}{\partial x}\right) - (\lambda + 2\mu) k_n u_y^p + K_{\text{atm}}\rho_0 g u_z^p = -h\sum_{j=0}^{p-1} \sigma_{yy}^j, \tag{15}$$

$$\frac{h}{2}\sigma_{zz}^p - \lambda\left(\frac{\partial u_x^p}{\partial x} + k_n u_y^p\right) - (\lambda + 2\mu)\frac{\partial u_z^p}{\partial z} + K_{\text{atm}}\rho_0 g u_z^p = -h\sum_{j=0}^{p-1} \sigma_{zz}^j, \tag{16}$$

$$\frac{h}{2}\sigma_{xy}^p - \mu\left(\frac{\partial u_y^p}{\partial x} + k_n u_x^p\right) = -h\sum_{j=0}^{p-1} \sigma_{xy}^j, \tag{17}$$

$$\frac{h}{2}\sigma_{xz}^p - \mu\left(\frac{\partial u_x^p}{\partial z} - \frac{\partial u_z^p}{\partial x}\right) = -h\sum_{j=0}^{p-1} \sigma_{xz}^j, \tag{18}$$

$$\frac{h}{2}\sigma_{yz}^p - \mu\left(\frac{\partial u_y^p}{\partial z} + k_n u_z^p\right) = -h\sum_{j=0}^{p-1} \sigma_{yz}^j, \tag{19}$$

$$K_{\mathrm{atm}}\left[\frac{h}{2}\rho^p + v_x\frac{\partial\rho^p}{\partial x} + \rho_0\left(\frac{\partial u_x^p}{\partial x} + k_n u_y^p + \frac{\partial u_z^p}{\partial z}\right) + u_z^p\frac{\partial\rho_0}{\partial z} = -h\sum_{j=0}^{p-1}\rho^j\right], \quad (20)$$

where f^p are the Laguerre coefficients of the source function $f(t)$. The coefficients u_x^p, u_y^p, u_z^p, σ_{xx}^p, σ_{yy}^p, σ_{zz}^p, σ_{xy}^p, σ_{xz}^p, σ_{yz}^p, ρ^p in formulas (11)–(20) are functions of the variables (n, x, z).

System (1)–(3) for the atmosphere is obtained from system (11)–(20) if we assume $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P$, $\mu = 0$, $\lambda = c_0^2\rho_0$, $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$, $K_{\mathrm{atm}} = 1$. Assuming in system (11)–(20) $K_{\mathrm{atm}} = 0$, we obtain the system of equations (4), (5) for the propagation of seismic waves in an elastic medium.

It is easy to see that the Laguerre transform parameter p is present only in the right-hand side of the equations and the spectral harmonics for all field components having a recurrent dependence. The condition of contact of two media at $z = 0$ is written down as

$$\begin{gathered} u_z^p|_{z=-0} = u_z^p|_{z=+0}, \\ \frac{h}{2}\sigma_{zz}^p + h\sum_{j=0}^{p-1}\sigma_{zz}^j\,\Big|_{z=-0} = \left(\frac{h}{2}\sigma_{zz}^p + h\sum_{j=0}^{p-1}\sigma_{zz}^j + \rho_0 g u_z^p\right)\Big|_{z=+0}, \\ \sigma_{xz}^p|_{z=-0} = \sigma_{yz}^p|_{z=-0} = 0. \end{gathered} \quad (21)$$

To solve the transformed problem (11)–(21), in contrast to the algorithm in [2, 3], we use the finite difference approximation of derivatives with respect to spatial coordinates on staggered grids with the 4th order of accuracy [9]. To do this, in the computational domain, we introduce in the direction z of the grid coordinates ωz_j and $\omega z_{j+1/2}$ with a sampling step Δz, staggered grids relative to each other by $\Delta z/2$:

$$\omega z_j = (x, j\Delta z), \quad \omega z_{j+1/2} = (x, j\Delta z + \Delta z/2), \quad j = 0, \ldots, M.$$

Similarly, we introduce in the direction x of the grid coordinates ωx_i and $\omega x_{i+1/2}$ with a sampling step Δx, staggered grids relative to each other by $\Delta x/2$:

$$\omega x_i = (i\Delta x, z), \quad \omega x_{i+1/2} = (i\Delta x + \Delta x/2, z), \quad i = 0, \ldots, K.$$

On these grids, we introduce the differentiation operators D_x and D_z, which approximate the derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$ with the fourth order of accuracy in the coordinates and

$$D_x u(x, z) = \frac{9}{8\Delta x}\Big[u(x + \Delta x/2, z) - u(x - \Delta x/2, z)\Big] - \frac{1}{24\Delta x}\Big[u(x + 3\Delta x/2, z) - u(x - 3\Delta x/2, z)\Big],$$

$$D_z u(x,z) = \frac{9}{8\Delta z}\Big[u(x, z + \Delta z/2) - u(x, z - \Delta z/2)\Big] - \frac{1}{24\Delta z}\Big[u(x, z + 3\Delta z/2) - u(x, z - 3\Delta z/2)\Big].$$

We define the sought-for components of the solution vector at the following grid nodes:

$$\begin{gathered} u_y^p(x,z),\ \ \sigma_{xx}^p(x,z),\ \ \sigma_{yy}^p(x,z),\ \ \sigma_{zz}^p(x,z),\ \ \rho^p(x,z) \quad \text{at } \omega x_i \times \omega z_j, \\ u_x^p(x,z),\ \ \sigma_{xy}^p(x,z) \quad \text{at } \omega x_{i+1/2} \times \omega z_j, \\ u_z^p(x,z),\ \ \sigma_{yz}^p(x,z) \quad \text{at } \omega x_i \times \omega z_{j+1/2}, \\ \sigma_{xz}^p(x,z) \quad \text{at } \omega x_{i+1/2} \times \omega z_{j+1/2}. \end{gathered}$$

As a result of the finite-difference approximation of problem (11)–(21), we obtain a system of linear algebraic equations. We represent the required solution vector $\vec{W}$ in the following form:

$$\begin{aligned} \vec{W}(p) &= (\vec{V}_{0,0}(p), \vec{V}_{0,1}(p), \ldots, \vec{V}_{M,K}(p))^T, \\ \vec{V}_{ij}(p) &= \Big(\rho_{xx}^p(x_i,z_j), \sigma_{xx}^p(x_i,z_j), \sigma_{yy}^p(x_i,z_j), \sigma_{zz}^p(x_i,z_j), \\ &\quad u_y^p(x_i,z_j), u_x^p(x_{i+1/2},z_j), \sigma_{xy}^p(x_{i+1/2},z_j), \\ &\quad u_z^p(x_i,z_{j+1/2}), \sigma_{yz}^p(x_i,z_{j+1/2}), \sigma_{xz}^p(x_{i+1/2},z_{j+1/2})\Big). \end{aligned}$$

Then, for each nth Fourier harmonic ($n = 0, \ldots, N$), the system of linear algebraic equations in the vector form can be written down as

$$\Big(A_\Delta + \frac{h}{2}E\Big)\vec{W}(p) = \vec{F}(p-1). \tag{22}$$

A sequence of the wave field components in the solution vector $\vec{V}$ is selected taking into account the minimization of the number of diagonals in the matrix A_Δ. In this case, on the main diagonal of the matrix, the components included in the equations of the system as terms with a parameter as a factor h (the parameter of the Laguerre transform) are specially located. It should be noted that due to the choice of the parameter h, it is possible to significantly improve the conditionality of the system matrix. Having solved the system of linear algebraic equations (22), one can determine the spectral values for all components of the wave field $\vec{W}(p)$. Then, using the inversion formulas for the Fourier transform (8), (9) and the Laguerre transform (10), we obtain a solution to the original problem.

3. Numerical results

Calculations of the wave field were applied for two models of the medium. The first model consists of a homogeneous elastic layer and an atmosphere separated by a flat boundary. The physical characteristics of the layers were set as follows:

1. The atmosphere: speed of sound $c_p = 340$ m/s. The density depending on the coordinate z was calculated by the formula $\rho_0(z) = \rho_1 \exp(-z/H)$, where $\rho_1 = 1.225 \cdot 10^{-3}$ g/cm^3, $H = 6700$ m;
2. The elastic layer: the longitudinal wave velocity $c_p = 800$ m/s, transverse wave velocity $c_s = 500$ m/s, the density $\rho_0 = 1.2$ g/cm^3.

For the calculations, we used a limited area of the medium with a dimension $80 \times 80 \times 40$ km^3. A wave field was simulated from a point source of the pressure center type located in an elastic medium at a depth of 1/4 of the longitudinal wave length with the coordinates $(x_0, y_0, z_0) = (40, 40, -0.2)$. The time signal in the sources was set in the form of the Puzyrev pulse:

$$f(t) = \exp\left(-\frac{2\pi f_0 (t-t_0)^2}{\gamma^2}\right) \sin(2\pi f_0 (t - t_0)), \tag{23}$$

where $\gamma = 4$, $f_0 = 1$ Hz, $t_0 = 1.5$ s.

Figure 1 shows the wavefield $u_x(x, y, z)$ snapshots for the in-plane XZ component at $y = y_0 = 40$ km. The interface between the elastic medium and the atmosphere is shown by a solid line. It can be seen from the figure that in an elastic medium, together with a spherical longitudinal wave P and a conical shear wave S, a "non-ray" spherical wave S^* propagates, followed by the surface Stoneley–Scholte wave R. Acoustic-gravity waves propagate in the atmosphere — the conical PP, SP and spherical P, followed by the Stoneley–Sholte surface wave.

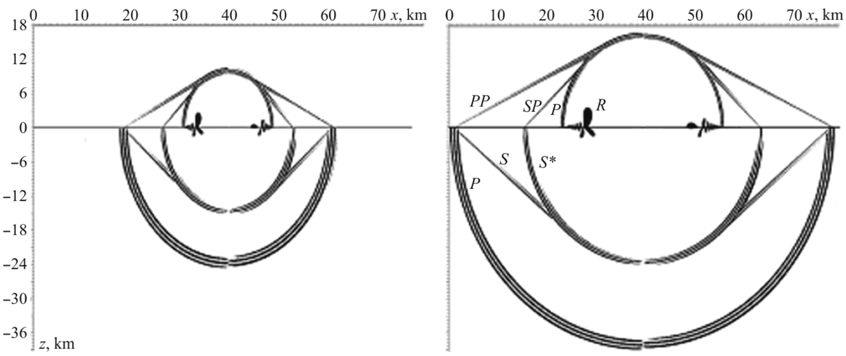

Figure 1. Snapshots of the horizontal velocity component u_x in the plane (XZ) at the time instant $t = 30$ (left) and 50 (right) seconds

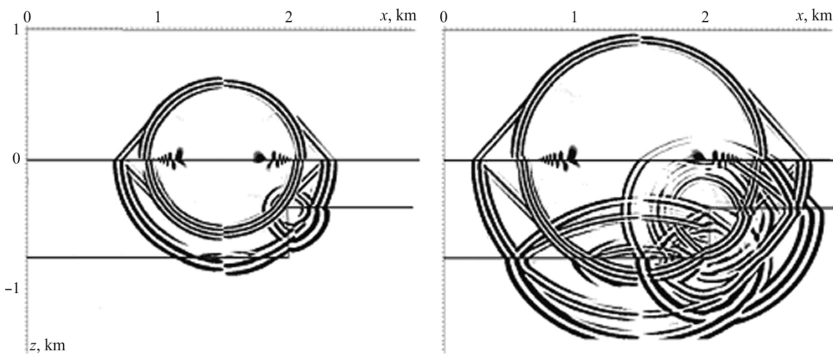

Figure 2. Snapshots of the horizontal velocity component u_x in the plane (XZ) at the time instant $t = 2$ (left) and 3 (right) seconds

Figure 2 shows snapshots of the wave field for the second model of the medium, in which the elastic medium consists of two layers, separated by a curved boundary in the form of a protrusion. The physical characteristics of the layers were set as follows:

1. The atmosphere: velocity of sound $c_p = 340$ m/s. The density depending on the height was calculated by the formula $\rho_0(z) = \rho_1 \exp(-z/H)$, where $\rho_1 = 1.225 \cdot 10^{-3}$ g/cm^3;
2. The upper elastic layer: the longitudinal wave velocity $c_p = 450$ m/s, the transverse wave velocity $c_s = 320$ m/s, the density $\rho_0 = 1.5$ g/cm^3;
3. The lower elastic layer: longitudinal wave velocity $c_p = 600$ m/s, the transverse wave velocity $c_s = 400$ m/s, the density $\rho_0 = 1.2$ g/cm^3.

A wave field was simulated from a point source of the vertical force type located in an elastic medium with the coordinates $(x_0, y_0, z_0) = (1.5, 1.5, -0.02)$. The time signal in the sources was specified by (23) for the frequency $f_0 = 8$ Hz.

From the snapshots of the wave field shown in the figure, it can be seen that seismic waves propagating in the lithosphere, falling on the curvilinear interface between the layers, generate the corresponding reflected and diffraction waves from this boundary.

Conclusion

The approach proposed to the formulation and solution of the considered problem allows one to simulate the effects of wave field propagation for a unified mathematical model of the "Earth–Atmosphere" medium and to study the processes of the appearance of converted waves at their interface. The numerical modeling of such processes makes it possible to study the features of the influence of the medium inhomogeneity on the propagation of acoustic-gravity and seismic waves and the occurrence of the Stoneley surface waves.

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