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Mixed vector finite element method for solving first order system of Maxwell equations*

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Abstract. This work is devoted to the construction and study of a computational scheme based on the mixed vector finite element method for modeling of the three-dimensional nonstationary electromagnetic fields. Numerical study of convergence of the mixed vector finite element method in the three-dimensional case on a class of problems, having a known smooth analytical solution is performed.

1. Introduction

The electromagnetic processes are described by a system of the Maxwell equations. When modeling electromagnetic fields, it is necessary to take into account the physical continuity conditions of electric and magnetic fields, typical of composite areas, where each of subregions has different physical characteristics. The problem of creation of efficient algorithms for the solution to the problems in electromagnetism is a real-life problem of computational mathematics.

Recently, alternative to the potential formulation of the problem is modeling of the electromagnetism problems in natural variables, i.e., writing down equations for the electric field intensity and the magnetic flux density or the magnetic field intensity. At the moment, the most wide-spread method for solving problems of electromagnetism is changing the system of the Maxwell equations to second order equations for the electric field ($\boldsymbol{E}$) or the magnetic field ($\boldsymbol{H}$), respectively [1–4]. The finite element method (FEM) and its modifications are widely used for solving the Maxwell equations system [5–7]. The FEM is a general method for the solution of differential equations [8]. To solve the second order equations in the natural variables $\boldsymbol{E}$ and $\boldsymbol{H}$, the vector FEM with a vector basis (edge elements), where degrees of freedom connected with the edges of a finite element net, is used. The edge elements provide continuity of a tangential component field on the inter-element and the inter-fragmentary borders in the area with discontinuous physical characteristics. Another approach is a direct solution to the system of first order equations on the basis of a mixed vector basis, consisting of edge elements and face elements, where degrees of freedom are connected with faces [9, 10].

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Face-elements realize continuity of a normal component of the magnetic flux density ($\boldsymbol{B}$). The mixed vector FEM allows solution of the electromagnetism problems in natural variables, with simultaneous obtaining values of electric and magnetic fields and with allowance for the physical characteristics of electromagnetic fields, such as continuity of a tangential component of the field ($\boldsymbol{E}$ or $\boldsymbol{H}$), or a normal component of the field ($\boldsymbol{D}$, $\boldsymbol{B}$).

In this paper, a mixed vector variational formulation for a system of the Maxwell equations in the time domain is constructed [11]. A mixed vector finite element analogue of the variational formulation is obtained on parallelepiped grids for edge- and face-basis functions. It is a systematic numerical research of the mixed vector FEM convergence in the three-dimensional case on a class of problems, having a known smooth analytical solution.

2. A mathematical model

We consider the system of the Maxwell equations, describing the behavior of the main electromagnetic field features:

$$\operatorname{rot}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \operatorname{rot}\boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \sigma\boldsymbol{E} + \boldsymbol{J}_0, \tag{1}$$

$$\operatorname{div}\boldsymbol{D} = \rho, \quad \operatorname{div}\boldsymbol{B} = 0. \tag{2}$$

The material equations are of the form

$$\boldsymbol{D} = \varepsilon\boldsymbol{E}, \quad \boldsymbol{B} = \mu\boldsymbol{H}, \tag{3}$$

where $\boldsymbol{E}$ is an electric field intensity, $\boldsymbol{D}$ is an electric flux density, $\boldsymbol{H}$ is a magnetic field intensity, $\boldsymbol{B}$ is a magnetic flux density, $\boldsymbol{J}_0$ is an external current density, σ is a specific conductivity, ρ is an electric charge density, ε is an electric permittivity, μ is a magnetic permeability.

Using correlations (3), and assuming $\mu = \text{const}$, we rewrite system (1), (2) as follows:

$$\operatorname{rot}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \mu^{-1}\operatorname{rot}\boldsymbol{B} = \varepsilon\frac{\partial \boldsymbol{E}}{\partial t} + \sigma\boldsymbol{E} + \boldsymbol{J}_0,$$
$$\operatorname{div}\boldsymbol{\varepsilon E} = \rho, \quad \operatorname{div}\boldsymbol{B} = 0.$$

We assume that the area, in which the electromagnetic field spreads, consists of subregions, values of the parameters ε and σ being given in each one. On the boundaries of the two subregions (Γ), the following conditions are given: $\boldsymbol{n} \times \boldsymbol{E}|_\Gamma = 0$.

The initial conditions and the boundary condition on the domain boundary are given as follows:

$$\boldsymbol{E}|_{t=t_0} = \boldsymbol{E}_0, \quad \boldsymbol{B}|_{t=t_0} = \boldsymbol{B}_0, \quad \boldsymbol{E} \times \boldsymbol{n}|_{\partial\Omega} = \boldsymbol{E}_g.$$

3. Variational formulation

We define the following function spaces

$$H(\mathrm{rot};\Omega) = \Big\{ \boldsymbol{v} \in [L_2(\Omega)]^3 : \ \mathrm{rot}\,\boldsymbol{v} \in [L_2(\Omega)]^3 \Big\},$$
$$H(\mathrm{div};\Omega) = \Big\{ \boldsymbol{v} \in [L_2(\Omega)]^3 : \ \mathrm{div}\,\boldsymbol{v} \in L_2(\Omega) \Big\},$$
$$H^0(\mathrm{rot};\Omega) = \Big\{ \boldsymbol{v} \in H(\mathrm{rot};\Omega) : \ \boldsymbol{v} \times \boldsymbol{n}|_{\partial\Omega} = 0 \Big\},$$
$$H^0(\mathrm{div};\Omega) = \Big\{ \boldsymbol{v} \in H(\mathrm{div};\Omega) : \ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \Big\},$$

where $\boldsymbol{n}$ is the external normal vector to the boundary $\partial\Omega$.

For incorporated spaces, the following inclusion conditions are valid

1. If $\varphi \in H(\Omega)$, then $\mathrm{grad}\,\varphi \in H(\mathrm{rot};\Omega)$.
2. If $\boldsymbol{E} \in H(\mathrm{rot};\Omega)$, then $\mathrm{rot}\,\boldsymbol{E} \in H(\mathrm{div};\Omega)$.

We define the inner product as follows:

$$(f, g) = \int_\Omega f \cdot g \, d\Omega.$$

Assuming the domain Ω does not have free charges ($\rho = 0$), it is possible to formulate the required problem for the electric and the magnetic fields:

For a given external current density $\boldsymbol{J}_0$ *such that* $\mathrm{div}\,\boldsymbol{J}_0 = 0$*, find* $\boldsymbol{E} \in H(\mathrm{rot};\Omega)$ *and* $\boldsymbol{B} \in H(\mathrm{div};\Omega)$ *such that*:

$$\mathrm{rot}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \text{ on } \Omega, \quad \mu^{-1}\,\mathrm{rot}\,\boldsymbol{B} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + \sigma \boldsymbol{E} + \boldsymbol{J}_0 \text{ on } \Omega, \tag{4}$$
$$\boldsymbol{E}|_{t=t_0} = \boldsymbol{E}_0, \quad \boldsymbol{B}|_{t=t_0} = \boldsymbol{B}_0, \quad \boldsymbol{E} \times \boldsymbol{n}|_{\partial\Omega} = \boldsymbol{E}_g. \tag{5}$$

Applying the scalar multiplication of functions from $H^0(\mathrm{div};\Omega)$ and $H^0(\mathrm{rot};\Omega)$ to the first and the last equations from (4), we arrive to the required variational formulation for problem (4), (5):

Find $\boldsymbol{E} \in H(\mathrm{rot};\Omega)$ *and* $\boldsymbol{B} \in H(\mathrm{div};\Omega)$ *such that for* $\forall\, \boldsymbol{V} \in H^0(\mathrm{rot};\Omega)$, $\forall\, \boldsymbol{F} \in H^0(\mathrm{div};\Omega)$

$$(\mathrm{rot}\,\boldsymbol{E}, \boldsymbol{F}) = -\frac{\partial}{\partial t}(\boldsymbol{B}, \boldsymbol{F}),$$
$$(\mu^{-1}\,\mathrm{rot}\,\boldsymbol{B}, \boldsymbol{V}) = \Big(\varepsilon \frac{\partial \boldsymbol{E}}{\partial t}, \boldsymbol{V}\Big) + (\sigma \boldsymbol{E} + \boldsymbol{J}_0, \boldsymbol{V}).$$

Using the vector identity $\mathrm{div}(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \mathrm{rot}\,\boldsymbol{a} - \boldsymbol{a} \cdot \mathrm{rot}\,\boldsymbol{b}$, we obtain

$$\int_\Omega \mathrm{rot}\,\mu^{-1}\boldsymbol{B} \cdot \boldsymbol{V} \, d\Omega = \int_\Omega \mathrm{rot}\,\boldsymbol{V} \cdot \mu^{-1}\boldsymbol{B} \, d\Omega + \int_\Omega \mathrm{div}\,(\boldsymbol{V} \times \mu^{-1}\boldsymbol{B}) \, d\Omega.$$

Using the Gauss–Ostrogradski formula

$$\int_\Omega \operatorname{div} u \, d\Omega = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} \, dS$$

and taking into account the definition of the test function $\boldsymbol{V}$, we have

$$\int_\Omega \operatorname{div} (\boldsymbol{V} \times \mu^{-1} \boldsymbol{B}) \, d\Omega = \int_{\partial\Omega} (\boldsymbol{V} \times \mu^{-1} \boldsymbol{B}) \cdot \boldsymbol{n} \, d\Omega = 0.$$

Then the final mixed variational formulation for the electric and the magnetic fields assumes the form:

Find $\boldsymbol{E} \in H(\mathrm{rot};\Omega)$ *and* $\boldsymbol{B} \in H(\mathrm{div};\Omega)$ *such that* $\forall \boldsymbol{V} \in H^0(\mathrm{rot};\Omega)$, $\forall \boldsymbol{F} \in H^0(\mathrm{div};\Omega)$

$$\int_\Omega \operatorname{rot} \boldsymbol{E} \cdot \boldsymbol{F} \, d\Omega = -\frac{\partial}{\partial t} \int_\Omega \boldsymbol{B} \cdot \boldsymbol{F} \, d\Omega, \tag{6}$$

$$\frac{\partial}{\partial t} \int_\Omega \varepsilon \boldsymbol{E} \cdot \boldsymbol{V} \, d\Omega = \int_\Omega \mu^{-1} \operatorname{rot} \boldsymbol{V} \cdot \boldsymbol{B} \, d\Omega - \int_\Omega (\sigma \boldsymbol{E} + \boldsymbol{J}_0) \cdot \boldsymbol{V} \, d\Omega. \tag{7}$$

Let us write down the condition of the magnetic field conservation as follows

$$\frac{\partial}{\partial t} (\mu^{-1} \boldsymbol{B}, \boldsymbol{F}) = (\mu^{-1} \operatorname{rot} \boldsymbol{E}, \boldsymbol{F}).$$

This is the projection $\operatorname{rot} \boldsymbol{E}$ on the space $H(\mathrm{div};\Omega)$. From the inclusion condition 2 it follows that this projection is exact (strict). Consequently, magnetic charges are preserved, i.e.,

$$\operatorname{div} \frac{\partial}{\partial t} \boldsymbol{B} = 0.$$

Since the electric field is broken, electric charges retained in the variational sense. The variational divergence has the form

$$\int_\Omega \varphi \operatorname{div} \varepsilon \boldsymbol{E} \, d\Omega = \int_\Omega \varepsilon \boldsymbol{E} \cdot \operatorname{grad} \varphi \, d\Omega.$$

Taking into account the inclusion condition 1, we obtain

$$\frac{\partial}{\partial t} (\varepsilon \boldsymbol{E}, \operatorname{grad} \varphi) = 0 \quad \forall \varphi \in H(\mathrm{rot};\Omega), \tag{8}$$

i.e., the electric field is orthogonal to all irrotational fields.

Let us write down Ampere's variational law

$$\Big(\varepsilon \frac{\partial \boldsymbol{E}}{\partial t}, \boldsymbol{V}\Big) = (\mu^{-1} \operatorname{rot} \boldsymbol{V}, \boldsymbol{B}). \tag{9}$$

Changing $\boldsymbol{V}$ by $\operatorname{grad} \varphi$, rewrite (9) in the form

$$(\varepsilon \frac{\partial \boldsymbol{E}}{\partial t}, \operatorname{grad} \varphi) = (\mu^{-1} \operatorname{rot} \operatorname{grad} \varphi, \boldsymbol{B}).$$

Taking $\operatorname{rot} \operatorname{grad} f \equiv 0$ into account, we come to the condition of the conservation of charge (8).

4. Discretization of variational formulation

Assume that the domain Ω is partitioned on a set of matched elements. Let us consider a parallelepiped element. We mark the lengths of the sides of the parallelepiped in the directions x, y, and z as l_x, l_y, and l_z, respectively. Let us mark the center of the parallelepiped as (x_c, y_c, z_c).

Define a discrete space $H^h(\mathrm{rot};\Omega) \subset H^0(\mathrm{rot};\Omega)$ entering basis functions on one element as follows (Figure 1):

$$\boldsymbol{N}_i = N_{xi}\boldsymbol{i}, \quad \boldsymbol{N}_{i+4} = N_{yi}\boldsymbol{j}, \quad \boldsymbol{N}_{i+8} = N_{zi}\boldsymbol{k} \quad \text{for } i = 1, 2, 3, 4.$$

Here $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ — unit orts,

$$N_{x1} = \frac{1}{l_y l_z}\Big(y_c+\frac{l_y}{2}-y\Big)\Big(z_c+\frac{l_z}{2}-z\Big), \quad N_{x2} = \frac{1}{l_y l_z}\Big(y-y_c+\frac{l_y}{2}\Big)\Big(z_c+\frac{l_z}{2}-z\Big),$$
$$N_{x3} = \frac{1}{l_y l_z}\Big(y_c+\frac{l_y}{2}-y\Big)\Big(z-z_c+\frac{l_z}{2}\Big), \quad N_{x4} = \frac{1}{l_y l_z}\Big(y-y_c+\frac{l_y}{2}\Big)\Big(z-z_c+\frac{l_z}{2}\Big),$$
$$N_{y1} = \frac{1}{l_x l_z}\Big(z_c+\frac{l_z}{2}-z\Big)\Big(x_c+\frac{l_x}{2}-x\Big), \quad N_{y2} = \frac{1}{l_x l_z}\Big(z-z_c+\frac{l_z}{2}\Big)\Big(x_c+\frac{l_x}{2}-x\Big),$$
$$N_{y3} = \frac{1}{l_x l_z}\Big(z_c+\frac{l_z}{2}-z\Big)\Big(x-x_c+\frac{l_x}{2}\Big), \quad N_{y4} = \frac{1}{l_x l_z}\Big(z-z_c+\frac{l_z}{2}\Big)\Big(x-x_c+\frac{l_x}{2}\Big),$$
$$N_{z1} = \frac{1}{l_y l_x}\Big(x_c+\frac{l_x}{2}-x\Big)\Big(y_c+\frac{l_y}{2}-y\Big), \quad N_{z2} = \frac{1}{l_y l_x}\Big(x-x_c+\frac{l_x}{2}\Big)\Big(y_c+\frac{l_y}{2}-y\Big),$$
$$N_{z3} = \frac{1}{l_y l_x}\Big(x_c+\frac{l_x}{2}-x\Big)\Big(y-y_c+\frac{l_y}{2}\Big), \quad N_{z4} = \frac{1}{l_y l_x}\Big(x-x_c+\frac{l_x}{2}\Big)\Big(y-y_c+\frac{l_y}{2}\Big).$$

We define a discrete space $H^h(\mathrm{div};\Omega) \subset H^0(\mathrm{rot};\Omega)$ determining basis functions on one element as follows (Figure 2):

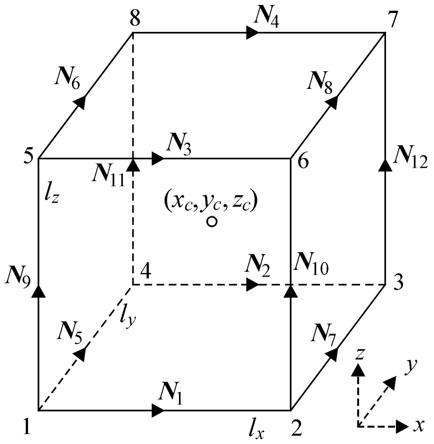

Figure 1. Edge functions

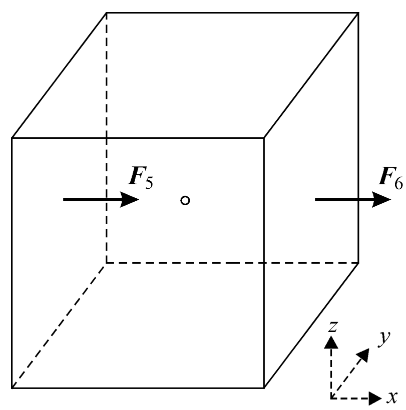

Figure 2. Face functions

$$\boldsymbol{F}_i = F_{zi}\boldsymbol{k}, \quad \boldsymbol{F}_{i+2} = F_{yi}\boldsymbol{j}, \quad \boldsymbol{F}_{i+4} = F_{xi}\boldsymbol{i} \quad \text{for } i = 1, 2.$$

Here

$$F_{z1} = \frac{1}{l_z}\Big(\frac{l_z}{2} + z_c - z\Big), \qquad F_{z2} = \frac{1}{l_z}\Big(\frac{l_z}{2} + z - z_c\Big),$$
$$F_{y1} = \frac{1}{l_y}\Big(\frac{l_y}{2} + y_c - y\Big), \qquad F_{y2} = \frac{1}{l_y}\Big(\frac{l_y}{2} + y - y_c\Big),$$
$$F_{x1} = \frac{1}{l_x}\Big(\frac{l_x}{2} + x_c - x\Big), \qquad F_{x2} = \frac{1}{l_x}\Big(\frac{l_x}{2} + x - x_c\Big).$$

The incorporated basis functions possess the following characteristics. The functions $\boldsymbol{N}_i$ guarantee the tangential continuity of electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ across the edges and the surfaces of the elements:

$$\operatorname{div} \boldsymbol{N}_i = 0, \quad \operatorname{rot} \boldsymbol{N}_i \neq 0.$$

The functions $\boldsymbol{F}_i$ guarantee the normal continuity of magnetic flux density $\boldsymbol{B}$ and electric flux density D across the faces of the elements:

$$\operatorname{rot} \boldsymbol{F}_i = 0, \quad \operatorname{div} \boldsymbol{F}_i \neq 0.$$

The functions $\boldsymbol{N}_i$ and $\boldsymbol{F}_i$ satisfy the inclusion condition 2. For instance,

$$\operatorname{rot} \boldsymbol{N}_1 = \operatorname{rot}\bigg(\frac{1}{l_y l_z}\Big(y_c + \frac{l_y}{2} - y\Big)\Big(z_c + \frac{l_z}{2} - z\Big)\boldsymbol{i}\bigg)$$
$$= -\frac{1}{l_y l_z}\bigg(\Big(y_c + \frac{l_y}{2} - y\Big)\boldsymbol{j} + \Big(z_c + \frac{l_z}{2} - z\Big)\boldsymbol{k}\bigg) = -\frac{1}{l_z}\boldsymbol{F}_3 - \frac{1}{l_y}\boldsymbol{F}_1.$$

The characteristics of the functions N_i and F_i allow us to conclude that they can be chosen as basis functions.

Using the discrete spaces $H^h(\text{rot};\Omega)$ and $H^h(\text{div};\Omega)$, we formulate discrete analogues of the variational formulations (6), (7):

Find $\boldsymbol{E} \in H^h(\text{rot};\Omega)$, $\boldsymbol{B} \in H^h(\text{div};\Omega)$ *such that* $\forall \boldsymbol{V} \in H^h(\text{rot};\Omega)$, $\forall \boldsymbol{F} \in H^h(\text{div};\Omega)$ *the equations* (6), (7) *hold.*

Let us seek the functions $\boldsymbol{E}$ and $\boldsymbol{B}$ on the basis of spaces $H^h(\text{rot};\Omega)$ and $H^h(\text{div};\Omega)$ in the form

$$\boldsymbol{E} = \sum_i \alpha_i(t)\boldsymbol{N}_i, \quad \boldsymbol{B} = \sum_j \beta_j(t)\boldsymbol{F}_j.$$

As a result, we obtain a coupled system of ordinary differential equations related to the factors $\alpha_i(t)$ and $\beta_j(t)$:

$$A\frac{d}{dt}\beta_j(t) = -K\alpha_i(t),$$
$$C\frac{d}{dt}\alpha_i(t) = K^T\beta_j(t) - M\alpha_i(t) - G.$$

Using the Rodrigue–White scheme for the time approximation, we arrive at the system of linear algebraic equations

$$\begin{aligned} &A\beta^{n+1/2} = A\beta^{n-1/2} - \Delta t K\alpha^n, \\ &\Big(C + \frac{\Delta t}{2}M\Big)\alpha^{n+1} = \Big(C - \frac{\Delta t}{2}M\Big)\alpha^n + \Delta t K^T\beta^{n+1/2} + \Delta t G^{n+1}, \end{aligned} \tag{10}$$

where entries of the matrices A, C, K, M and the vector of the right-hand side G are defined as follows:

$$[A]_{ij} = \int_\Omega \mu^{-1}\boldsymbol{F}_i \cdot \boldsymbol{F}_{\boldsymbol{j}}\, d\Omega, \qquad [C]_{ij} = \int_\Omega \varepsilon \boldsymbol{N}_i \cdot \boldsymbol{N}_j\, d\Omega,$$

$$[M]_{ij} = \int_\Omega \sigma \boldsymbol{N}_i \cdot \boldsymbol{N}_j\, d\Omega, \qquad [K]_{ij} = \int_\Omega \mu^{-1}\boldsymbol{F}_i \cdot \boldsymbol{N}_j\, d\Omega,$$

$$[G]_i = \int_\Omega \boldsymbol{J}_0 \cdot \boldsymbol{N}_i\, d\Omega + \int_{\Gamma_2} (\boldsymbol{N}_i \times \mu^{-1}\boldsymbol{B}) \cdot \boldsymbol{n}\, d\Gamma.$$

Let us define local matrices for the vector basis functions

$$[A]^e_{ij} = \int_{\Omega^e} \mu^{-1}\boldsymbol{F}^e_i \cdot \boldsymbol{F}^e_j\, d\Omega, \quad [R]^e_{ij} = \int_{\Omega^e} \boldsymbol{N}^e_i \cdot \boldsymbol{N}^e_j\, d\Omega,$$

$$[K]^e_{ij} = \int_{\Omega^e} \mu^{-1}\boldsymbol{F}^e_i \cdot \boldsymbol{N}^e_j\, d\Omega,$$

where Ω_e is a finite element.

As a result, we obtain a specific structure of the local matrices

$$(K^e)^T = \frac{\mu^{-1}}{6}\begin{bmatrix} 2l_xl_z & l_xl_z & -2l_xl_y & -l_xl_y & 0 & 0 \\ -2l_xl_z & -l_xl_z & -l_xl_y & -2l_xl_y & 0 & 0 \\ l_xl_z & 2l_xl_z & 2l_xl_y & l_xl_y & 0 & 0 \\ -l_xl_z & -2l_xl_z & l_xl_y & 2l_xl_y & 0 & 0 \\ -2l_yl_z & -l_yl_z & 0 & 0 & 2l_xl_y & l_xl_y \\ -l_yl_z & -2l_yl_z & 0 & 0 & -2l_xl_y & -l_xl_y \\ 2l_yl_z & l_yl_z & 0 & 0 & l_xl_y & 2l_xl_y \\ l_yl_z & 2l_yl_z & 0 & 0 & -l_xl_y & -2l_xl_y \\ 0 & 0 & 2l_yl_z & l_yl_z & -2l_xl_z & -l_xl_z \\ 0 & 0 & -2l_yl_z & -l_yl_z & -l_xl_z & -2l_xl_z \\ 0 & 0 & l_yl_z & 2l_yl_z & 2l_xl_z & l_xl_z \\ 0 & 0 & -l_yl_z & -2l_yl_z & l_xl_z & 2l_xl_z \end{bmatrix},$$

$$A^e = \frac{l_xl_yl_z}{6}\mu^{-1}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad R^e = \begin{bmatrix} R_{xx} & 0 & 0 \\ 0 & R_{yy} & 0 \\ 0 & 0 & R_{zz} \end{bmatrix},$$

$$[R_{pp}]_{kl} = \int_{\Omega^e} N_{pk} N_{pl}\, d\Omega, \quad R_{pp} = \frac{l_x l_y l_z}{36} \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix}, \quad p \in \{x, y, z\}.$$

5. Numerical experiments

The numerical investigation of the mixed vector FEM was carried out on problems (4), (5) of modeling of a nonstationary electromagnetic field in the domain $\Omega = [0,1]^3$, $t \geq 0$. Testing was carried out on the known analytical solution:

$$\boldsymbol{E} = \begin{bmatrix} x^2 y \\ -2xy^2 \\ 2xyz \end{bmatrix} e^{-\alpha t}, \quad \boldsymbol{B} = \frac{1}{\alpha} \begin{bmatrix} 2xz \\ -2yz \\ -2y^2 - x^2 \end{bmatrix} e^{-\alpha t}.$$

The exterior current density corresponding to the analytical solution has the form

$$\boldsymbol{J}_0 = \frac{1}{\alpha\mu} \begin{bmatrix} 2y \\ -4x \\ 0 \end{bmatrix} e^{-\alpha t} + (\sigma - \varepsilon\alpha) \begin{bmatrix} x^2 y \\ -2xy^2 \\ 2xyz \end{bmatrix} e^{-\alpha t},$$

where $\sigma = 1$ (Om·m)$^{-1}$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $\alpha = 10^7$.

In the table below, the results of the numerical solution of the problem on a number of nested grids are presented. We use here the following notations: N_i^E and N_i^B are the dimensions of spaces $H^h(\mathrm{rot};\Omega)$ and $H^h(\mathrm{div};\Omega)$, respectively; h is a spatial in all the three directions; Δt is a time step; ΔE and ΔB are the relative errors of the approximate solution for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ respectively at the point $(0.4, 0.4, 0.4)$ at the time moment $t = 2.01 \cdot 10^{-10}$ s; $\Delta E = |\boldsymbol{E} - \tilde{\boldsymbol{E}}|/|\boldsymbol{E}|$, $\Delta B = |\boldsymbol{B} - \tilde{\boldsymbol{B}}|/|\boldsymbol{B}|$, $\boldsymbol{E}$ and $\boldsymbol{B}$ are the analytic solutions, $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{B}}$ are the numerical solutions.

On every time step, the resulting systems of linear algebraic equations (SLAE) are solved by the conjugate gradient method, the condition of the solver exit being reduction of the ratio error in 10^{14} times. Dimensions of the SLAE are provided in the table.

The result of the given study verifies stability and convergence of the presented computational scheme on smooth solutions.

Results of modeling on a number of nested grids

N_i^E	N_i^B	h	Δt	ΔE	ΔB
540	450	0.2	$2 \cdot 10^{-12}$	$4.658 \cdot 10^{-2}$	$4.548 \cdot 10^{-2}$
3630	3300	0.1	$1 \cdot 10^{-12}$	$1.165 \cdot 10^{-2}$	$1.137 \cdot 10^{-2}$
26460	25200	0.05	$5 \cdot 10^{-13}$	$2.913 \cdot 10^{-3}$	$2.842 \cdot 10^{-3}$

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