Bull. Nov. Comp. Center, Math. Model. in Geoph., 14 (2011), 59–64

Modeling of the free convection in a viscous compressible fluid

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Convection is one of the basic types of the flows used in description of processes of the Earth's interior and the Earth's atmosphere. The simulation of convection of many geological systems under the assumption of an incompressible medium in the Boussinesq approximation often appears to be poor and the model of a compressible medium is used not only when simulating processes in the atmosphere, but, also, when simulating dense media of such geological systems as magmatic fluid systems, the convective upper mantle with allowance for phase transitions, dynamics of magma in deep chambers and magma-conductors, a convective warm-up of the lithosphere magmatic fluids, dynamics of liquid hydrothermal systems with allowance for magma boiling and hydrothermal solution, condensation of a gas, etc. In the given paper, a numerical model for analysis of dynamics of such media with variable kinetic and thermodynamic parameters is constructed. The choice of a numerical algorithm was determined by the necessity to ensure the physical correctness of the solution at arbitrary spatial and time scales of the system under investigation as a large spatial scale of the system and large times of its evolution make it difficult to carry out laboratory simulation as well as field measurements. The control volume method satisfies this condition.

1. Problem formulation

In this paper, the problem about the free convection of a compressible fluid in the classical statement [1], which is used in many papers, presenting a "benchmark solution" [2–4] is studied. The domain of the square form (Figure 1) is investigated. On the side boundaries of the computational domain, the values of temperature are fixed: on the left-hand wall, the value of temperature T_H is given higher than the value of temperature T_C on the right-hand wall. The upper and the bottom walls are considered to be adiabatic. On all the boundaries, conditions of non-flow and slippages are set. As was mentioned above, a fluid is taken to be Newtonian, viscous, compressible. As the objective of calculations here is deriving a stationary flow, the assignment representation of the initial distribution of the basic values is not of first importance. The temperature at the initial moment is constant and equal to the temperature on a cold wall (T_C), and the medium is considered to be at rest.

The system of governing equations, defining a convective flow of a fluid, includes the equation of continuity, the Navier–Stokes equation, and the entropy equation, respectively:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0,\\
&\rho\frac{\partial \boldsymbol{u}}{\partial t} + (\rho\boldsymbol{u},\nabla)\boldsymbol{u} = -\nabla p + \frac{\partial}{\partial x_k}\left[\eta\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\operatorname{div}\boldsymbol{u}\right)\right] + \rho\mathbf{g}, \qquad (1)\\
&\rho\frac{\partial s}{\partial t} + \rho(\boldsymbol{u},\nabla)s = \varkappa\frac{1}{T}\Delta T.
\end{aligned}$$

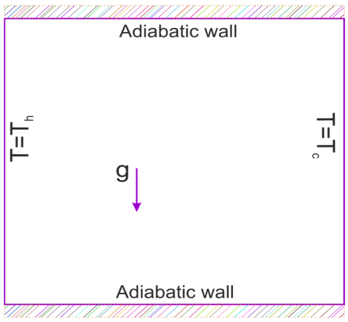

Figure 1. Statement of the problem about the natural convection

In the system of equations (1) and further the variables $\boldsymbol{u}$, ρ, p, T, and s denote velocity, density, pressure, temperature, and entropy, respectively; η is the dynamic viscosity, $\nu = \eta/\rho$ is the kinematic viscosity, $\varkappa$ is the heat conductivity, $\chi = \varkappa/(c_p\rho)$ is the thermal conductivity; and $\mathbf{g}$ is the gravitational acceleration.

The equations of state close the system:

$$\begin{aligned}
T &= T_0 + e_{\rho s}(\rho - \rho_0) + e_{ss}(s - s_0),\\
p &= p_0 + \rho_0^2 e_{\rho\rho}(\rho - \rho_0) + \rho_0^2 e_{\rho s}(s - s_0),
\end{aligned} \qquad (2)$$

in which the coefficients e_{ss}, $e_{\rho s}$, $e_{\rho\rho}$ are expressed through the values of known thermodynamic parameters of the medium: the thermal expansion coefficient β, the compressibility α, the heat capacity c_p:

$$e_{\rho\rho} = \frac{c_p}{Td}, \quad e_{ss} = \rho_0\frac{\alpha}{d}, \quad e_{\rho s} = \rho_0\frac{\beta}{d},$$

Here, $d = \rho_0\alpha c_p/T - \rho_0^2\beta^2$, the parameters ρ_0, s_0, $T_0 = T_c$, $p_0 = \rho_0\mathbf{g}\boldsymbol{r}$ characterize an arbitrary non-stress state of continuum. In this paper, an assumption about constant coefficients of the equation of state, and, also, the values of dynamic viscosity, thermal conductivity, heat capacity is made.

Let us reduce system (1), (2) to the dimensionless form. As thermal processes are defining, we use dimensionless time in terms of thermal conductivity χ, that is the characteristic time of the system is determined by thermal processes:

$$\begin{aligned}
t = \frac{l^2}{\chi}\tilde{t}, \quad x = l\tilde{x}, \quad y = l\tilde{y}, \quad \boldsymbol{u} = \frac{\chi}{l}\tilde{\boldsymbol{u}}, \quad p - p_0 = \rho_0\frac{\chi^2}{l^2}\tilde{p},\\
T = \Delta T(k + \widetilde{T}), \quad k = \frac{T_c}{\Delta T}, \quad \Delta T = T_h - T_c.
\end{aligned} \qquad (3)$$

The dimensionless system of equations (1) takes the form

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0, \\ &\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}, \nabla)\boldsymbol{u} = -\nabla p + \Pr \Delta \boldsymbol{u} - \mathbf{e}\,\mathrm{Ra}\Pr T + \mathbf{e} N_1 p, \\ &\frac{\partial T}{\partial t} + (\boldsymbol{u}, \nabla) T = \Delta T + N_2 \frac{\partial p}{\partial t} + N_2 (\boldsymbol{u}, \nabla) p \end{aligned} \tag{4}$$

and is characterized by the dimensionless parameters

$$\mathrm{Ra} = \frac{g\beta\Delta T l^3}{\nu\chi}, \quad \Pr = \frac{\nu}{\chi}, \quad N_1 = \mathbf{g}\alpha\rho_0 l, \quad N_2 = \frac{\beta k \chi^2}{c_p l^2},$$

where Ra is the Rayleigh number, the basic similarity criterion when studying the free convection of an incompressible medium; Pr is the Prandtl number; N_1 and N_2 are dimensionless numbers being a corollary of compressibility of a fluid in the system taken into account (4). Hereafter, all parameters are dimensionless and the tilde is omitted.

The numerical solution of boundary value problem is realized on the basis of the control volume method (CVM) [5] which provides the physical behavior of approximations of the governing equations and hence the solutions of the problem on any time scale. This is a necessary condition for investigation of processes in the geological systems characterized by large spatial scales and large time periods of the development of processes, whose experimental and field verification is difficult or even impossible. An important property of the CVM is a precise integral conservation of such quantities, as mass, impulse and energy in any group of control volumes and, hence, in the whole computational domain.

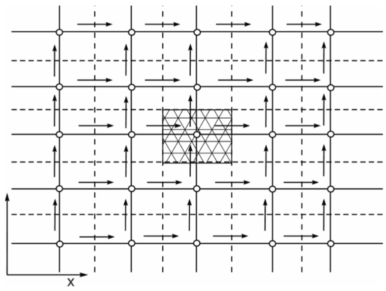

Figure 2. A chess grid and appropriate notations

It is convenient to make an approximation of differential equations of motion of a fluid in the problem about the free convection on a staggered uniform grid (Figure 2). At the node points $\circ$ the values of ρ, P, T, s and at the node points $\rightarrow$, $\uparrow$ the values of vertical and horizontal components of the velocity (u, v) are respectively calculated.

When applying the numerical algorithm in question to the solution of problems associated with the analysis of geological models for constructing discrete analogs of the differential equations, the time implicit approximation scheme is used. This provides the possibility of carrying out calculations on large time intervals, that is, on geological time intervals. For calculation of values of independent variables on the boundaries of control volumes, the upstream difference scheme that improves the physical behavior of a discrete analog and results in acceleration of the convergence rate is used.

For calculation of the pressure field and the flow field at each time step, the iteration scheme SIMPLE [5] was used, which consists in the following: the initial pressure field is given; discrete analogs of the equations of motion are solved and an approximate value of the velocity is determined; the equation for determination of grid values with allowance for pressure is solved; the corrected field of pressure is calculated; the corrected value of the velocity from the corrector formulas and from approximate values is found at Step 1; the new value of density from the equation of state with allowance for the corrected pressure field is calculated; the corrected pressure is represented as a new approximate value of the pressure field and the procedure is repeated up to attaining the convergence of the solution. Further, a discrete analog of the entropy equation is solved, from the equation of state the temperature is calculated and passage to the next time step is made.

The convergence criterion of the iteration scheme SIMPLE is considered to be the discrepancy of the continuity equation. The linear algebraic equation systems obtained at discretization of governing equations, will be solved with the use of the alternating direction method.

2. Results and discussions

When solving a test problem, we investigated the convection of water with the following properties under normal conditions ($P_0 = 1.01325 \cdot 10^5\,\mathrm{Pa}$, $T_0 = 293$ K): the density $\rho_0 = 999.8$ kg/m^3; the dynamic viscosity, the thermal conduction and the specific heat of the medium are determined by the following values, respectively: $\mu = 0.0017888$ kg/(s m), $\varkappa = 0.566$ W/(K m), $c_p = 4212.0$ J/(kg K). Coefficients of isothermal dilatation and volumetric compression also correspond to parameters of water: $\beta = -9.973486 \cdot 10^{-5}$ K^{-1}, $\alpha = 2.493870 \cdot 10^{-7}$ Pa^{-1}.

Calculations were conducted on the grid with discretization up to 81×81 nodes on each of spatial coordinates. It was assumed that the computational domain is of the square form.

For the comparison of the numerical results obtained with those represented in [2], the calculations were carried out with the respective Rayleigh numbers: Ra $= 10^3$, 10^5, 10^7. The comparison was made using values of the local Nusselt number on the hot boundary $\mathrm{Nu} = \left.\frac{\partial T}{\partial x}\right|_{x=0}$ whose graphs are represented in [2]. Results of calculations of temperature fields and their comparison with those in [2] are presented in Figure 3 (a) and (b). The comparisons of the calculated modification of the local Nusselt number along the hot boundary with standard results obtained by other authors are presented in Figure 3 (c).

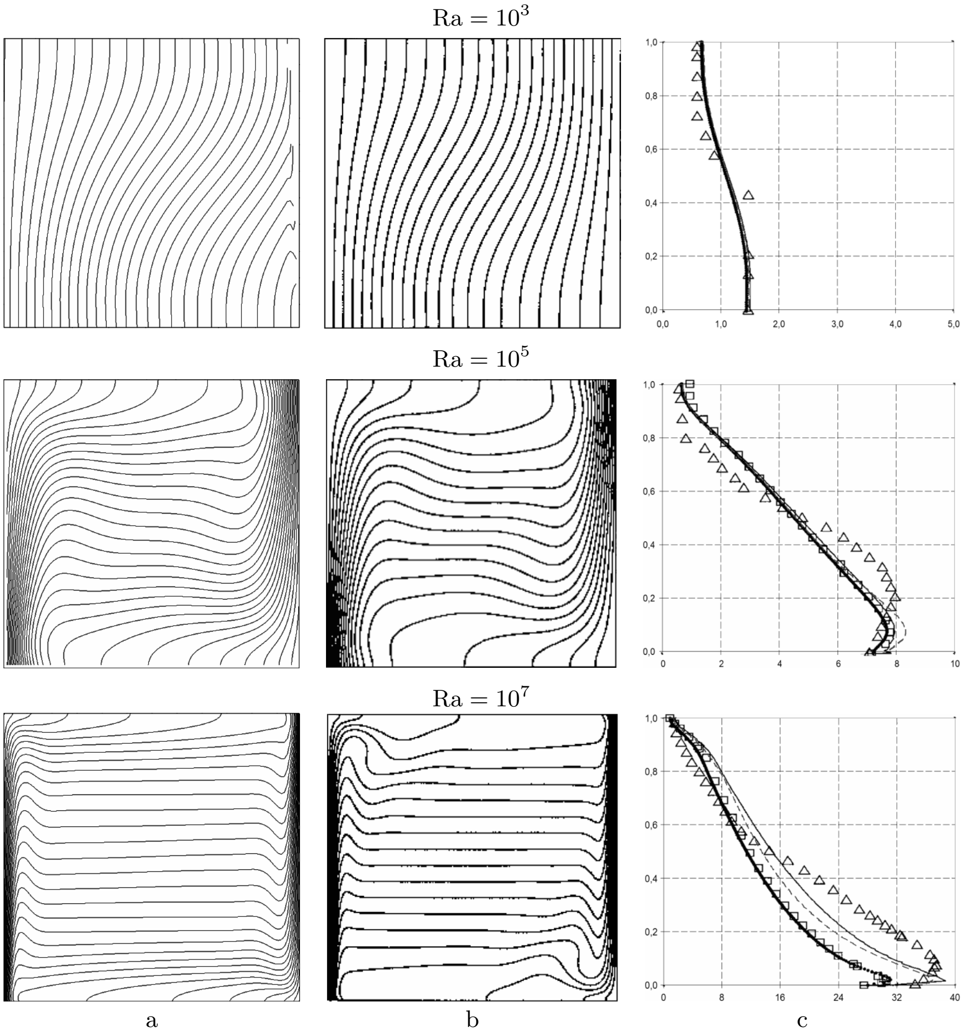

Figure 3. The comparison of temperature distribution: (a) the results of the given paper, (b) the benchmark [2], and (c) the local Nusselt number Nu vs. y-coordinate (— [2], △ [3], □ [4], **—** FEM [2], – – MCV)

At $\mathrm{Ra} = 10^3$, is a weakly intensive flow with low gradients of temperature and pressure. A very good coincidence of the calculated temperature distribution and, especially, of the local Nusselt number with standard results [2] is distinct, which means the absence of the influence of the compressibility of a fluid on the picture of a flow and on the defining conductive mechanism of the heat in the system transfer with these parameters.

At $\mathrm{Ra} = 10^5$, the flow becomes more intensive, the heat transfer begins to be defined by a convective flow, but a good coincidence of the results with calculations for an incompressible liquid is conserved.

At Ra $= 10^7$, an intensive flow with high gradients of temperature and pressure is developing. At such high Rayleigh numbers, the compressibility of a fluid starts to affect the development of a convective flow. In the represented figures, a difference between the distribution of the local Nusselt number and standard results, which is associated with allowance for compressibility, is observed.

Thus, in this paper, the numerical model is obtained that permits carrying out the simulation of the heat-mass transfer processes in compressible media with variable viscosity. Differences at large Rayleigh numbers denote the range of parameters in which the compressibility effects start to be of first importance. The given model will be further developed in the two-velocity model of convection in the explored geological systems. The computer model obtained can be used as a hydrodynamic basis when simulating the evolution of magmatic, hydrothermal systems, when investigating the ultrasonic action on a convective the flow in fluid and in the fluid-magmatic systems.

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