Bull. Nov. Comp. Center, Math. Model. in Geoph., 10 (2005), 81–91

Wave equation inversion with redundance data by the wave field continuation method*

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Abstract. The inverse problem of wave scattering for a system of data collected at the observation surface with multiple coverage of sources and receivers is considered. It is shown that this redundant data system is equivalent to the configuration studied previously, in which the receiver point coincides with the source point: in both cases the wave field continuation back in time reduces the scattering problem considered in a linearized approximation to the Cauchy problem (the one with instantaneously actuated secondary field sources represented by the sought-for inhomogeneities) in the reference medium with a "half" value of wave velocity.

1. Introduction

In this paper, a linearized statement of the inverse dynamic problem as related to the wave equation for an observation system with multiple coverage of sources and receivers is considered. In seismic prospecting, such systems are widespread and generally considered to be effective for digital processing of field data, signal extraction against the background of noise, etc. From the point of view of information, such systems are clearly redundant for solving the inverse problem of scattering. For instance, if oscillations of observation surface points recorded for all possible locations of a point source are considered as initial data for the solution of the inverse problem, the input data take the form of a five-dimensional function (two parameters that determine the receiver location, two parameters that determine the source location, and time). However, in this case, the three-dimensional velocity function must be reconstructed.

Therefore, it is interesting to find out how this data redundancy behaves at the inversion of the wave equation. The continuation of the field is used to solve this problem, but then the field is continued from the observation surface (both from receivers and sources) into the region under investigation. This possibility is in the wave equation as it is, for which the reciprocity principle is realized in a simple form [1].

One can obtain the most appropriate description of the field continuation process from the observation surface into the region in question and back in time to solve the inverse problem of reconstructing field sources by using the Green identity, which is well-known in mathematical physics. In the case

*Supported by the Russian Foundation for Basic Research under Grant 03-05-64081.

under consideration, the medium inhomogeneities play the role of secondary sources. There is only one distinctive feature owing to which the use of the Green identity in the problem of the wave field continuation differs from the conventional rule of its use in problems of mathematical physics, that the employment of the Green function (more precisely, the fundamental solution) of the *advanced* type [2–4]:

$$\begin{aligned} &\Box G_+(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = \delta(\boldsymbol{x}-\boldsymbol{\xi})\delta(t-\tau) \\ &\qquad\qquad \text{for } \boldsymbol{x}\in\mathbb{R}^3,\ t\in\mathbb{R}^1;\ \boldsymbol{\xi}\in D_0,\ \tau>0; \\ &G_+(\boldsymbol{x},\boldsymbol{\xi},t-\tau)\equiv 0 \qquad \text{for } \boldsymbol{x}\in\mathbb{R}^3,\ t>\tau. \end{aligned} \tag{1.1}$$

Here, $\Box$ is a wave operator acting at the field point $(\boldsymbol{x},t)$, $(\boldsymbol{\xi},\tau)$ are parameters; D_0 is a finite domain in $\mathbb{R}^3$ (which is of interest for the investigation) bounded by the curvilinear surface S_0.

If the inequality sign in the second condition from (1.1) is changed to the opposite one, we determine the well-known ("causal") Green function of the *retarded* type:

$$G_-(\boldsymbol{x},\boldsymbol{\xi},t-\tau)\equiv 0 \qquad \text{for } t<\tau, \tag{1.2}$$

From the standpoint of calculation, however, a linear combination of (1.1) and (1.2) is most useful:

$$G_*(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = G_-(\boldsymbol{x},\boldsymbol{\xi},t-\tau) - G_+(\boldsymbol{x},\boldsymbol{\xi},t-\tau). \tag{1.3}$$

The fundamental solution G_* is defined as a solution to the following Cauchy problem in the entire four-dimensional space $\mathbb{R}^{3+1}$ [5, 6]:

$$\begin{aligned} &\Box G_*(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = 0 \quad \text{for } \boldsymbol{x}\in\mathbb{R}^3,\ t\in\mathbb{R}^1;\ \boldsymbol{\xi}\in D_0,\ \tau>0; \\ &G_*(\boldsymbol{x},\boldsymbol{\xi},0)=0, \quad \frac{\partial}{\partial t}G_*(\boldsymbol{x},\boldsymbol{\xi},0) = -c^2\delta(\boldsymbol{x}-\boldsymbol{\xi}) \quad \text{for } \boldsymbol{x}\in\mathbb{R}^3,\ t=\tau. \end{aligned} \tag{1.4}$$

In a homogeneous medium characterized by the velocity $c_0 = \text{const}$, the solution G_* has the following explicit form [1]:

$$\begin{aligned} G_*(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = -\frac{1}{4\pi}\biggl[&|\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\delta\biggl(t-\tau-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0}\biggr) - \\ &|\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\delta\biggl(t-\tau+\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0}\biggr)\biggr]. \end{aligned} \tag{1.5}$$

It should be noted that G_* is the odd function with respect to t:

$$G_*(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = -G_*(\boldsymbol{x},\boldsymbol{\xi},\tau-t),$$

and G_+ is a mirror reflection of G_- onto the negative semi-axis $t<0$:

$$G_+(\boldsymbol{x},\boldsymbol{\xi},t-\tau) = G_-(\boldsymbol{x},\boldsymbol{\xi},\tau-t).$$

The advantage of using G_* instead of G_+ in the continuation procedures is that the field continued with the help of G_* is smoother, does not contain any discontinuities of the field and its normal derivative at the surface S_0, which inevitably take place when using of the function G_+ (Appendix A). At the same time, G_* includes all properties of the *advanced* function G_+ that are necessary for the wave field inversion.

2. Statement of the problem

The following statement of the wave scattering problem in an inhomogeneous medium is considered as basic model:

$$\Big(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)U(\boldsymbol{x}, t; \boldsymbol{z}) = \delta(\boldsymbol{x} - \boldsymbol{z})\delta(t) \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}^1;\ \boldsymbol{z} \in S_0; \tag{2.1}$$

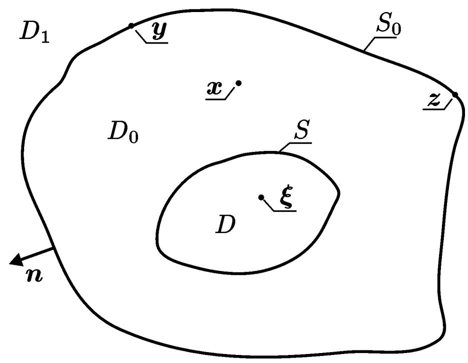
$$U(\boldsymbol{x}, t; \boldsymbol{z}) \equiv 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t < 0.$$

Here, the variable velocity $c(\boldsymbol{x})$ is assumed to be different from *const* only in the local domain D (the figure). From here on, the coefficient $c^{-2} \equiv c^{-2}(\boldsymbol{x})$ in equation (2.1) is represented as sum of the two terms:

$$c^{-2}(\boldsymbol{x}) = c_0^{-2} + m(\boldsymbol{x}), \tag{2.2}$$

where the velocity $c_0 = \text{const}$ is considered to be known (the reference model), and the velocity anomaly carrier $\overline{D} \equiv \operatorname{supp} m(\boldsymbol{x})$ represents a compact that is fully located in the domain under investigation, $D \Subset D_0$.

It is also assumed that the variable velocity $c(\boldsymbol{x})$ has sufficient smoothness, so that there are no boundaries with discontinuous changes in the wave velocity. Therefore, there is no need to set additional boundary conditions of contact between the media (however, such conditions can, in principle, be removed, and is introduced only to simplify the consideration).



Let the wave process be realized according to conditions (2.1), and its trace, together with the normal derivative, be "observed" at the surface S_0. The inverse problem of scattering is to reconstruct the three-dimensional velocity anomaly $m(\boldsymbol{x})$ with the use of this information.

It should be noted that the union $D_0 \cup S_0 \cup D_1$ forms $\mathbb{R}^3$, and problem (2.1) is set for the entire space $\mathbb{R}^3$. The surface S_0 (see the figure) is a virtual surface, which is used, as indicated above, to "record" the field and its normal derivative. Therefore, no boundary conditions are set. This refined statement is taken specially to simplify further reasoning and, mainly, to call attention to the conceptual aspect of the problem of the wave equation inversion in an idealized situation of maximally possible information about the source wave field. In fact, before using more realistic statements, one should answer the following question: what can be said about the source of the field of scattered waves, if this field, together with the normal derivative, is known at all points of the closed surface surrounding the scattering object?

According to the idea of linearization, the full field U from (2.1) is represented as sum of the two terms:

$$U(\boldsymbol{x}, t; \boldsymbol{z}) = U_{\text{in}}(\boldsymbol{x}, t; \boldsymbol{z}) + u(\boldsymbol{x}, t; \boldsymbol{z}), \tag{2.3}$$

in which the incident field (a sounding signal) U_{in} is due to the action of a concentrated source in the reference medium with the velocity c_0 (without anomaly):

$$\begin{aligned} &\Big(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\Big)U_{\text{in}}(\boldsymbol{x}, t; \boldsymbol{z}) = \delta(\boldsymbol{x} - \boldsymbol{z})\delta(t) \\ &\qquad\qquad\qquad\qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}^1;\ \boldsymbol{z} \in S_0; \\ &U_{in}(\boldsymbol{x}, t; \boldsymbol{z}) \equiv 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t < 0. \end{aligned} \tag{2.4}$$

If we assume that $\boldsymbol{\xi} = \boldsymbol{z}$, $\tau = 0$, the incident field U_{in} practically coincides with fundamental solution (1.2) of the wave operator with a constant velocity, and has the following form [1, 7]:

$$U_{\text{in}}(\boldsymbol{x}, t; \boldsymbol{z}) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{z}|}\delta\Big(t - \frac{|\boldsymbol{x} - \boldsymbol{z}|}{c_0}\Big), \tag{2.5}$$

which is different from zero for $t > 0$ and equal to zero for $t < 0$.

The scattered field $u(\boldsymbol{x}, t; \boldsymbol{z})$ is determined by the conditions:

$$\begin{aligned} &\Big(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\Big)u(\boldsymbol{x}, t; \boldsymbol{z}) = m(\boldsymbol{x})\frac{\partial^2}{\partial t^2}U_{\text{in}}(\boldsymbol{x}, t; \boldsymbol{z}) \\ &\qquad\qquad\qquad\qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t > 0;\ \boldsymbol{z} \in S_0; \\ &u(\boldsymbol{x}, t; \boldsymbol{z}) \equiv 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t < 0. \end{aligned} \tag{2.6}$$

The linearization problem is in the fact that the right-hand side of equation (2.6) has the incident field U_{in} instead of the full field U, which should be present at a true transition from (2.1) to (2.6).

Further simplification is obtained if the convolution operator acts on both parts of equality (2.6) only in the time domain with the function t_+ determined by the following relation:

$$t_+ = tH(t) = H(t) * H(t), \tag{2.7}$$

where $H(t)$ is the Heaviside function, and the symbol "$*$" denotes the convolution operation.

As a result, instead of (2.6) we obtain for the transformed field

$$v(\boldsymbol{x}, t; \boldsymbol{z}) = t_+ * u(\boldsymbol{x}, t; \boldsymbol{z}) \tag{2.8}$$

the following conditions:

$$\begin{aligned} &\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) v(\boldsymbol{x}, t; \boldsymbol{z}) = m(\boldsymbol{x}) U_{\text{in}}(\boldsymbol{x}, t; \boldsymbol{z}) \\ &\qquad\qquad\qquad\qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3, \ t > 0; \ \boldsymbol{z} \in S_0; \\ &v(\boldsymbol{x}, t; \boldsymbol{z}) \equiv 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3, \ t < 0. \end{aligned} \tag{2.9}$$

The character of transformation (2.8) becomes clear if we take into account the spectral form of the function t_+ in the frequency domain [7]:

$$F[t_+] = i\pi\delta'(\omega) - \frac{1}{\omega^2}, \tag{2.10}$$

where $F[t_+]$ denotes the Fourier transform of the function t_+.

Transformation (2.8) can be treated as a sort of filtration, in whose process low-frequency components are intensified in the observation data. This, in turn, reveals a general low-frequency nature of the Born scattering, within which the inverse problem is considered. It should also be noted that the delta function at zero frequency included in (2.10) does not practically "act", since the frequency characteristic of a seismic channel with whose help the wave field is recorded tends at $\omega = 0$ to zero stronger than ω^2.

3. Solution of the problem

Now, let us apply the Green identity (A.1) to two functions, namely, to v from (2.9) and G_* from (1.4). As a result, for $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}^1$, $\boldsymbol{z} \in S_0$ we have

$$w(\boldsymbol{x}, t; \boldsymbol{z}) = -\iiint_D m(\boldsymbol{\xi}) G_*(\boldsymbol{x}, \boldsymbol{\xi}, t) * U_{in}(\boldsymbol{\xi}, t; \boldsymbol{z})\, dV_\xi, \tag{3.1}$$

where w denotes the field continued (at receivers) from the surface S_0, formed with the surface integral in the Green formula:

$$w(\boldsymbol{x}, t; \boldsymbol{z}) = \iint_{S_0} \Big[v_0(\boldsymbol{y}, t; \boldsymbol{z}) * \frac{\partial}{\partial n_y} G_*(\boldsymbol{x}, \boldsymbol{y}, t) - \mu(\boldsymbol{y}, t; \boldsymbol{z}) * G_*(\boldsymbol{x}, \boldsymbol{y}, t) \Big] dS_y, \tag{3.2}$$

We use here the following notations:

$$\boldsymbol{y} \equiv \boldsymbol{x} \in S_0; \quad v_0(\boldsymbol{y}, t; \boldsymbol{z}) \equiv v(\boldsymbol{x} \in S_0, t; \boldsymbol{z}); \quad \mu(\boldsymbol{y}, t; \boldsymbol{z}) \equiv \frac{\partial v}{\partial n_x}(\boldsymbol{x} \in S_0, t; \boldsymbol{z}),$$

and the symbol "$*$" in (3.1) and (3.2) denotes the time convolution operation.

Owing to the properties of the fundamental solution G_*, the field w continued with its help is smooth and satisfies the homogeneous wave equation:

$$\Box w(\boldsymbol{x}, t; \boldsymbol{z}) = 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3, \ t \in \mathbb{R}^1, \ \boldsymbol{z} \in S_0. \tag{3.3}$$

The wave operator in (3.3) acts at the field point $(\boldsymbol{x}, t)$, and $\boldsymbol{z}$ is a parameter. It should also be noted that as $\boldsymbol{x} \to S_0$, the field w_0 does not coincide with v_0, and their normal derivatives at points of the surface S_0 are also different.

Now let us continue both parts of equality (3.1) at the source points $\boldsymbol{z} \in S_0$. As a result, for $\boldsymbol{x} \in \mathbb{R}^3$, $t \in \mathbb{R}^1$ we obtain

$$W(\boldsymbol{x}, t) = \iiint\limits_D m(\boldsymbol{\xi}) G_*(\boldsymbol{x}, \boldsymbol{\xi}, t; c_0) * G_*(\boldsymbol{x}, \boldsymbol{\xi}, t; c_0) \, dV_\xi, \tag{3.4}$$

where formula (A.2) is used, and the continued field W is given by a surface integral (at a source point) in the Green formula:

$$W(\boldsymbol{x}, t) = \iint_{S_0} \left[w_0(\boldsymbol{x}, t; \boldsymbol{z}) * \frac{\partial}{\partial n_z} G_*(\boldsymbol{x}, \boldsymbol{z}, t; c_0) - \gamma(\boldsymbol{x}, t; \boldsymbol{z}) * G_*(\boldsymbol{x}, \boldsymbol{z}, t; c_0) \right] dS_z, \tag{3.5}$$

where $\gamma(\boldsymbol{x}, t; \boldsymbol{z}) \equiv \partial_{n_z} w(\boldsymbol{x}, t; \boldsymbol{z})$, and the symbol "$*$" denotes, as before, the time convolution operation.

Owing to the reciprocity property of the fundamental solution $G_*(\boldsymbol{x}, \boldsymbol{\xi}, t)$ concerning the points $\boldsymbol{x}$ and $\boldsymbol{\xi}$, the continued field W (at sources and receivers) is smooth and satisfies the homogeneous wave equation:

$$\Box W(\boldsymbol{x}, t) = 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3, \ t \in \mathbb{R}^1, \tag{3.6}$$

and as $\boldsymbol{x} \to S_0$, the field W_0 does not coincide with w_0, and their normal derivatives at points of the surface S_0 are also different.

Let us transform (only with respect to time) the continued field W according to the following rule:

$$V(\boldsymbol{x}, t) = \frac{8\pi}{c_0} t \, W(\boldsymbol{x}, t). \tag{3.7}$$

Now, instead of (3.4), taking into account formula (B.2), we obtain

$$V(\boldsymbol{x},t) = -\left(\frac{2}{c_0}\right)^2 \iiint_D m(\boldsymbol{\xi}) G_*(\boldsymbol{x},\boldsymbol{\xi},t;c_0/2)\,dV_\xi. \tag{3.8}$$

The field $V(\boldsymbol{x},t)$ is the solution to the following Cauchy problem in the medium with the velocity $c_0/2$:

$$\begin{aligned} &\left(\Delta - \frac{4}{c_0^2}\frac{\partial^2}{\partial t^2}\right) V(\boldsymbol{x},t) = 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}^1; \\ &V(\boldsymbol{x},0) = 0, \qquad \frac{\partial}{\partial t} V(\boldsymbol{x},0) = m(\boldsymbol{x}). \end{aligned} \tag{3.9}$$

The last equality in (3.9) gives the solution to the inverse problem of scattering: it reconstructs the velocity anomaly $m(\boldsymbol{x})$ and, hence, the velocity $c(\boldsymbol{x})$ according to (2.2).

4. Analysis of the solution

To compare the results, let us consider the inverse problem of scattering (2.1) for the data at S_0 collected by the observation system, in which the source point coincides with the receiver, that is, $\boldsymbol{y} = \boldsymbol{z}$ (see the figure on page 83). In this case, repeating the reasoning of the previous section, we find that the transformation of the scattered field in the time domain, which reduces the scattering problem to the Cauchy problem, has the following form:

$$v(\boldsymbol{x},t) = \frac{8\pi}{c_0}\frac{\partial}{\partial t}\Big\{ t\big[t_+ * u(\boldsymbol{x},t;\boldsymbol{x})\big]\Big\}. \tag{4.1}$$

Now, the field $v(\boldsymbol{x},t)$, instead of (2.9), is determined by the conditions

$$\begin{aligned} &\left(\Delta - \frac{4}{c_0^2}\frac{\partial^2}{\partial t^2}\right) v(\boldsymbol{x},t) = -\frac{4}{c_0^2} m(\boldsymbol{x})\delta'(t) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}^1; \\ &v(\boldsymbol{x},t) \equiv 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ t < 0. \end{aligned} \tag{4.2}$$

Assuming that the field trace (4.1) and its normal derivative at S_0 are known, we continue this field into the medium with the velocity $c_0/2$:

$$\begin{aligned} w(\boldsymbol{x},t) = \iint_{S_0} \Big[v_0(\boldsymbol{y},t) * \frac{\partial}{\partial n_y} G_*(\boldsymbol{x},\boldsymbol{y},t;c_0/2) \,- \\ \mu(\boldsymbol{y},t) * G_*(\boldsymbol{x},\boldsymbol{y},t;c_0/2)\Big]\,dS_y. \end{aligned} \tag{4.3}$$

The following notations were used here:

$$\boldsymbol{y} \equiv \boldsymbol{x} \in S_0; \quad v_0(\boldsymbol{y}, t) \equiv v(\boldsymbol{x} \in S_0, t); \quad \mu(\boldsymbol{y}, t) \equiv \frac{\partial v}{\partial n_x}(\boldsymbol{x} \in S_0, t),$$

and the symbol "$*$" in (4.1) and (4.3) denotes the time convolution operation.

Owing to the properties of the fundamental solution G_*, the field w of (4.3) continued with its help is smooth everywhere in $\mathbb{R}^{3+1}$ and satisfies the following conditions:

$$\begin{aligned} &\Big(\Delta - \frac{4}{c_0^2}\frac{\partial^2}{\partial t^2}\Big) w(\boldsymbol{x}, t) = 0 \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^3, \ t \in \mathbb{R}^1; \\ &w(\boldsymbol{x}, 0) = m(\boldsymbol{x}), \qquad \frac{\partial}{\partial t} w(\boldsymbol{x}, 0) = 0. \end{aligned} \tag{4.4}$$

A comparison of (3.9) and (4.4) shows that the field continuation in both cases reduces, in fact, the scattering problem to the same Cauchy problem in the medium with the velocity $c_0/2$. The only difference is that in the case of data redundancy, the field is continued from the surface S_0 into the medium both at receivers and sources with the velocity c_0, whereas in the case of the observation system, in which the source coincides with the receiver, the field is continued from the surface S_0 into the medium with the velocity $c_0/2$.

Solution to the inverse problem of scattering for the observation system, in which the source coincides with the receiver, was obtained earlier in a more realistic statement in [8–10]. A plane, at whose points the second boundary condition (an analogue to the "free surface" in elasticity theory) must be satisfied was taken as the observation surface S_0. The solution to this problem has the form of (4.1)–(4.4) up to the factor "2" (due to the influence of the "free surface"). This indicates to the fact that the restrictions associated with the closure of the observation surface and with the need for "measuring" the normal derivative of the field, prove to be insignificant and can be removed. However, the restrictions, associated with the problem linearization, present in all these cases, are of a fundamental nature, and still cannot be removed.

It should also be noted that the linearized inverse problem of wave scattering for the data system, in which the source point coincides with the receiver point, can be reduced to the problem of integral geometry of the function reconstruction by using its spherical averages, for which the theorem of solution uniqueness has been proved [11].

From the physical point of view, the full field of oscillations (2.3) can be separated into a sounding signal and the medium response in a simple and, what is more important, more reliable way, only by observing the field near a source of oscillations, and at other surface S_0 points, such a separation is an unclear procedure. However, from the information standpoint, the field continuation by using both receivers and sources leads to the Cauchy problem, which is obtained if to solve the inverse problem we use only data with $\boldsymbol{y} = \boldsymbol{z}$; the other data simply do not play any role in the wave field inversion. Therefore, even if there are certain advantages of the system with multiple coverage, they exist in the paradigm "signal/noise" which, in the case of the wave scattering problem, again has an unclear physical nature (because of an unclear definition of a "signal" in this case).

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A. The Green identity

Let the functions u and G satisfy the wave equation in $\mathbb{R}^{3+1}$, and let the function G (see (1.1)–(1.5)) be a fundamental solution. Then the Green identity can be written down in the following form:

$$\kappa\, u(\boldsymbol{x},t) - \iiint_{D_0} \Box u(\boldsymbol{\xi},t) * G(\boldsymbol{x},\boldsymbol{\xi},t)\, dV_\xi$$
$$= \iint_{S_0} \Big[u(\boldsymbol{\xi},t) * \frac{\partial}{\partial n_\xi} G(\boldsymbol{x},\boldsymbol{\xi},t) - \partial_n u(\boldsymbol{\xi},t) * G(\boldsymbol{x},\boldsymbol{\xi},t) \Big] dS_\xi, \quad \text{(A.1)}$$

where

$$\kappa = \begin{cases} 1 & \text{for } \boldsymbol{x} \in D_0, \\ 0 & \text{for } \boldsymbol{x} \notin \overline{D}_0, \\ 0.5 & \text{for } \boldsymbol{x} \in S_0 \end{cases}$$

(see the figure on page 83).

Any of the solutions G_-, G_+ can be taken as function G. If G_* is used, then $\kappa = 0$ everywhere for $\boldsymbol{x} \in \mathbb{R}^3$. It should be noted that the above Green formula is written in the form including the initial conditions of the function u: they are transferred into the right-hand side of the wave operator $\Box u$ in the form of instantaneously actuated sources (which can always be done) [7]. The symbol "$*$" in (A.1) denotes the time convolution operation.

Expression (A.1) clearly shows a discontinuous character of the field determined from the fundamental solutions G_-, G_+, and smooth properties of the field formed with the help of the fundamental solution G_*.

If we take a pair of functions, $u \equiv U_{in}$ and $G \equiv G_*$, the Green formula gives the following representation of the function G_*: for $\boldsymbol{x}, \boldsymbol{\xi} \in \mathbb{R}^3$, $t \in \mathbb{R}^1$

$$- G_*(\boldsymbol{x},\boldsymbol{\xi},t) = \iint_{S_0} \Big[U_{\text{in}}(\boldsymbol{x},t;\boldsymbol{z}) * \frac{\partial}{\partial n_z} G_*(\boldsymbol{\xi},\boldsymbol{z},t) - \frac{\partial}{\partial n_z} U_{in}(\boldsymbol{x},t;\boldsymbol{z}) * G_*(\boldsymbol{\xi},\boldsymbol{z},t) \Big] dS_z. \quad \text{(A.2)}$$

B. Formulas for G_*

To calculate the time convolution of the function G_* with itself, we use its explicit expression (1.5). As a result, we obtain

$$G_*(\boldsymbol{x},\boldsymbol{\xi},t;c_0) * G_*(\boldsymbol{x},\boldsymbol{\xi},t;c_0) = \frac{1}{16\pi^2} \Big[|\boldsymbol{x}-\boldsymbol{\xi}|^{-2} \delta\Big(t - \frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0/2}\Big) + |\boldsymbol{x}-\boldsymbol{\xi}|^{-2} \delta\Big(t + \frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0/2}\Big) - 2|\boldsymbol{x}-\boldsymbol{\xi}\|^{-2}\delta(t) \Big]. \quad \text{(B.1)}$$

Let us multiply both parts of equality (B.1) by t and use the known properties of delta functions [7]:

$$t\delta(t) = 0, \qquad f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau).$$

As a result, we have

$$\frac{1}{16\pi^2}t\Big[|\boldsymbol{x}-\boldsymbol{\xi}|^{-2}\delta\Big(t-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0/2}\Big)+|\boldsymbol{x}-\boldsymbol{\xi}|^{-2}\delta\Big(t+\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0/2}\Big)\Big]$$
$$=\frac{1}{16\pi^2}\frac{2}{c_0}\Big[|\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\delta\Big(t-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0/2}\Big)-|\boldsymbol{x}-\boldsymbol{\xi}|^{-1}\delta\Big(t+\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c_0/2}\Big)\Big]$$
$$=-\frac{1}{2\pi c_0}G_*(\boldsymbol{x},\boldsymbol{\xi},t;c_0/2)$$

Thus, we obtain the following formula:

$$t\big\{G_*(\boldsymbol{x},\boldsymbol{\xi},t;c_0)*G_*(\boldsymbol{x},\boldsymbol{\xi},t;c_0)\big\}=-\frac{1}{2\pi c_0}G_*(\boldsymbol{x},\boldsymbol{\xi},t;c_0/2). \tag{B.2}$$

This formula makes it possible to reduce the scattering problem (in the medium with the velocity c_0) in the Born approximation for a system with multiple coverage of receivers and sources to the Cauchy problem in the medium with the velocity $c_0/2$.

Owing to the properties of the fundamental solution G_* (1.4), from representation (B.2) we finally obtain the following formula:

$$\frac{\partial}{\partial t}\Big\{t\big[G_*(\boldsymbol{x},\boldsymbol{\xi},t;c_0)*G_*(\boldsymbol{x},\boldsymbol{\xi},t;c_0)\big]\Big\}\Big|_{t\to 0}=\frac{c_0}{8\pi}\delta(\boldsymbol{x}-\boldsymbol{\xi}). \tag{B.3}$$

This formula solves the inverse problem of scattering under consideration for the system with multiple coverage of receivers and sources.