

Suspension hydrodynamic equations: two-phase models*

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Abstract. Two-velocity models of suspension hydrodynamics are considered, using the equality condition for chemical potentials of the phases or without this condition. The model equations are obtained by the conservation law method and are thermodynamically consistent under the assumption of one pressure or two pressure systems. Models are compared numerically on the test problems of discontinuity decay using the TVD method. A comparison was also made with a two-velocity suspension model with an internal energy dependent on the phase volume fractions.

Keywords: two-phase media, two-velocity hydrodynamics, conservation law method, hyperbolic systems, TVD method

Introduction

This paper discusses models of hydrodynamics for two-velocity media suitable for study the mechanics of liquid mixtures, granular materials, bubbly liquids, and suspensions. The models under study were obtained by the phenomenological method of conservation laws [1, 2] which ensures the thermodynamic consistency of the model and the hyperbolicity of the equations in the reversible approximation. The property of hyperbolicity for the systems of equations makes it possible to use effective numerical methods and to study the system behavior on discontinuous solutions. In this paper, the analysis of equations is based on the explicit GFORCE method [3–5]. The GFORCE method was widely developed in [6–9], as well as difference schemes based on the TVD method [6]. The advantage of the TVD-scheme, based on the principle of non-increase of the total variation of the solution, is the monotonicity property of solution. The aim of this paper is to study the influence of the given thermodynamic properties of a two-phase medium model on the relative phase dynamics that determines the choice of a specific model for describing natural and technology systems.

1. Mathematical models of two-phase media

Model 1. The dynamic properties of continuous media models are determined by the functional dependence of the internal energy of the medium

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under consideration. In this model, we will assume the following dependence of the internal energy (per unit volume) of a two-phase medium $E_0 = E_0(\rho, S, \mathbf{j}_0)$

$$dE_0 = \mu d\rho + T dS + (\mathbf{w}, d\mathbf{j}_0).$$

Then the pressure can be determined by the relation

$$p = -E_0 + \mu\rho + TS + (\mathbf{w}, \mathbf{j}_0).$$

The system of two-velocity medium motion equations with equilibrium of pressures for every phase (obtained by the method of conservation laws) takes the following form [10]

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \mathbf{u}_1) = 0, \quad \frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 \mathbf{u}_2) = 0, \quad (1)$$

$$\frac{\partial j_i}{\partial t} + \partial_k(\rho_1 u_{1i} u_{1k} + \rho_2 u_{2i} u_{2k} + p \delta_{ik}) = 0, \quad (2)$$

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u}_1, \nabla) \mathbf{w} + (\mathbf{w}, \nabla) \mathbf{u}_2 + \frac{1}{2} \nabla w^2 + b \frac{\rho}{\rho_1} \mathbf{w} = 0, \quad (3)$$

$$\frac{\partial S}{\partial t} + \operatorname{div}\left(\frac{S}{\rho} \mathbf{j}\right) = \frac{R}{T}. \quad (4)$$

This system of equations fulfills the energy conservation law

$$\frac{\partial E}{\partial t} + \operatorname{div}\left((p + E) \frac{\mathbf{j}}{\rho} + \frac{\rho_2}{\rho} (\mathbf{u}_1, \mathbf{j}_0) \mathbf{w}\right) = 0. \quad (5)$$

Here ρ_1, ρ_2 are the partial phase densities, $\rho = \rho_1 + \rho_2$ is the density of the two-phase medium, $\mathbf{u}_1, \mathbf{u}_2$ are the phase velocities, $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ is the relative phase velocity, $\mathbf{j} = \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_2$ is the momentum of the two-phase medium, $\mathbf{j}_0 = \rho_1 \mathbf{w}$ is the relative momentum, p is the pressure, T is the temperature, μ is the chemical potential, S is the entropy density per unit volume, and b is the coefficient of interfacial friction. Since the model takes into account the energy dissipation only due to interfacial friction, the dissipative function R has a simple form $R = b\rho_2 w^2$.

The energy density per unit volume is given by the relation

$$E = E_0 + \rho_1 (\mathbf{u}_2, \mathbf{w}) + \frac{1}{2} \rho u_2^2. \quad (6)$$

The system of equations (1)–(4) is closed by setting the equation of state

$$p = p(\rho, S, w^2), \quad T = T(\rho, S, w^2).$$

The fractional flow equations for such a medium have the form

$$\begin{aligned}\frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_1, \nabla) \mathbf{u}_1 &= -\frac{1}{\rho} \nabla p - \frac{\rho_2}{2\rho} \nabla w^2 - b \frac{\rho_2}{\rho_1} \mathbf{w}, \\ \frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_2, \nabla) \mathbf{u}_2 &= -\frac{1}{\rho} \nabla p + \frac{\rho_1}{2\rho} \nabla w^2 + b \mathbf{w}.\end{aligned}$$

Model 2. In this model, the equality of the chemical potentials between the phases will not be accepted, as a result, the pressures in the phases will also be different. The dependence of the specific internal energy of such a two-phase medium will be as follows: $E_0 = E_0(\rho, \rho_1, S, \mathbf{j}_0)$,

$$dE_0 = \mu d\rho + q d\rho_1 + T dS + (\mathbf{w}, d\mathbf{j}_0).$$

Then the pressure can be determined by the relation

$$p = -E_0 + \mu\rho + q\rho_1 + TS + (\mathbf{w}, \mathbf{j}_0).$$

The system of motion equations for a two-velocity medium has the form [11]:

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \mathbf{u}_1) = 0, \quad \frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \mathbf{u}_1) = 0, \quad (7)$$

$$\frac{\partial j_i}{\partial t} + \partial_k (\rho_1 u_{1i} u_{1k} + \rho_2 u_{2i} u_{2k} + p \delta_{ik}) = 0, \quad (8)$$

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u}_1, \nabla) \mathbf{w} + (\mathbf{w}, \nabla) \mathbf{u}_2 + \nabla q + \frac{1}{2} \nabla w^2 + b \frac{\rho}{\rho_1} \mathbf{w} = 0, \quad (9)$$

$$\frac{\partial S}{\partial t} + \operatorname{div} \left(\frac{S}{\rho} \mathbf{j} \right) = \frac{R}{T}. \quad (10)$$

The consequence of the system of equations (7)–(10) is the energy conservation law (5). The energy density per unit volume is determined by relation (6).

Above, the interfacial interaction parameter q has been defined; this parameter, which along with the pressure, determines the pressures in the phases p_1 and p_2 .

The system of equations (7)–(10) is closed by setting the equation of state

$$p = p(\rho, \rho_1, S, w^2), \quad T = T(\rho, \rho_1, S, w^2).$$

The motion equations for this medium have the form

$$\begin{aligned}\frac{\partial \mathbf{u}_1}{\partial t} + (\mathbf{u}_1, \nabla) \mathbf{u}_1 &= -\frac{1}{\rho} \nabla p - \frac{\rho_2}{\rho} \nabla q - \frac{\rho_2}{2\rho} \nabla w^2 - b \frac{\rho_2}{\rho_1} \mathbf{w}, \\ \frac{\partial \mathbf{u}_2}{\partial t} + (\mathbf{u}_2, \nabla) \mathbf{u}_2 &= -\frac{1}{\rho} \nabla p + \frac{\rho_1}{\rho} \nabla q + \frac{\rho_1}{2\rho} \nabla w^2 + b \mathbf{w}.\end{aligned}$$

2. Difference scheme

The systems of equations of two-velocity hydrodynamics Model 1 (1)–(4) and Model 2 (7)–(10) in the one-dimensional case are represented as discrete analogs of the conservation laws

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = Q. \quad (11)$$

For integration over time, a first-order scheme is used in a space-time cell $[x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}]$ with steps $\Delta x = x_{i+1/2} - x_{i-1/2}$ and $\Delta t = t^{n+1} - t^n$:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} (F_{i+1/2} - F_{i-1/2}) + \Delta t Q,$$

where flows are calculated according to the GFORCE scheme using a combination of Lax–Friedrichs and Lax–Wendroff flows [6,9]

$$F_{i+1/2}^{\text{GF}} = \theta F_{i+1/2}^{\text{LW}} + (1 - \theta) F_{i+1/2}^{\text{LF}}, \quad 0 \leq \theta \leq 1,$$

$$F_{i+1/2}^{\text{LF}} = \frac{1}{2} (F(U_i^n) + F(U_{i+1}^n)) - \frac{1}{2} \frac{\Delta x}{\Delta t} (U_{i+1}^n - U_i^n), \quad (12)$$

$$F_{i+1/2}^{\text{LW}} = F(U_{i+1/2}^{\text{LW}}), \quad U_{i+1/2}^{\text{LW}} = \frac{1}{2} (U_i^n + U_{i+1}^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(U_{i+1}^n) - F(U_i^n)). \quad (13)$$

The time step is calculated according to the formula

$$\Delta t = K_{\text{CFL}} \min_i \frac{\Delta x}{C_i^{\text{max}}},$$

where K_{CFL} is the Courant number, C_i^{max} is the maximum speed of sound.

The GFORCE method is used in conjunction with TVD reconstruction with a minmod limiter. Flows (12) and (13) are calculated as follows

$$F_{i+1/2}^{\text{LF}} = \frac{1}{2} (F(U_L^n) + F(U_R^n)) - \frac{1}{2} \frac{\Delta x}{\Delta t} (U_R^n - U_L^n),$$

$$F_{i+1/2}^{\text{LW}} = F(U_{i+1/2}^{\text{LW}}), \quad U_{i+1/2}^{\text{LW}} = \frac{1}{2} (U_L^n + U_R^n) - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(U_R^n) - F(U_L^n)),$$

where

$$U_L^n = U_i^n - \frac{1}{2} \sigma_i, \quad U_R^n = U_i^n + \frac{1}{2} \sigma_i,$$

$$\sigma_i = \begin{cases} \max(0, \min(\Delta_{i-1/2}, \Delta_{i+1/2})), & \Delta_{i+1/2} \geq 0, \\ \min(0, \max(\Delta_{i-1/2}, \Delta_{i+1/2})), & \Delta_{i+1/2} < 0, \end{cases}$$

$$\Delta_{i-1/2} = U_i^n - U_{i-1}^n, \quad \Delta_{i+1/2} = U_{i+1}^n - U_i^n.$$

3. Model results comparison

The models were compared using the problem of discontinuity decay in the one-dimensional isentropic approximation taken as an example. The initial data for the first phase were taken as $\rho_{01}^{\text{ph}} = 1000 \text{ kg/m}^3$, $\gamma_1 = 2.8$, $C_{01} = 1540 \text{ m/s}$, $c_{V1} = 4.2 \text{ kJ/(kg}\cdot\text{K)}$, $\alpha_1 = 0.89$, $u_{01} = 0 \text{ m/s}$, for the second phase $\rho_{02}^{\text{ph}} = 0.66 \text{ kg/m}^3$, $\gamma_2 = 1.4$, $C_{02} = 430 \text{ m/s}$, $c_{V2} = 0.7 \text{ kJ/(kg}\cdot\text{K)}$, $\alpha_2 = 0.11$, $u_{02} = 0 \text{ m/s}$. At the initial time in the middle of the region, the pressure discontinuity is set for each of the components:

$$p_i(0, x) = \begin{cases} 10^6 \text{ Pa}, & 0 \leq x \leq 0.5, \\ 10^5 \text{ Pa}, & 0.5 \leq x \leq 1, \end{cases} \quad i = 1, 2, \dots$$

The equation of state in these calculations was chosen as the van der Waals equation

$$p_1(\rho_1) = p_{01} + \rho_{01} \frac{C_{01}^2}{\gamma_1} \left(\left(\frac{\rho_1}{\rho_{01}} \right)^{\gamma_1} - 1 \right), \quad p_2(\rho_2) = p_{02} \frac{C_{02}^2}{\gamma_2} \left(\frac{\rho_2}{\rho_{02}} \right)^{\gamma_2},$$

where ρ_{0i} are the physical densities under normal conditions.

The results of the comparison are presented in Figures 1 as a comparison of the pressure and velocity profiles for the second phase at time $t = 1.85 \text{ ms}$.

Figures 1a and 1b demonstrate the close behavior of the pressures of the models with one and two pressures, while the distribution of phase velocities differs significantly. We will demonstrate this on a problem in which the pressure jump in each of the phases will be different. Let us carry out a numerical comparison with the initial data for the first phase $\rho_{01}^{\text{ph}} = 998 \text{ kg/m}^3$, $\gamma_1 = 3.0$, $C_{01} = 1407 \text{ m/s}$, $u_{01} = 0 \text{ m/s}$ and for the second phase $\rho_{02}^{\text{ph}} = 850 \text{ kg/m}^3$, $\gamma_2 = 2.96$, $C_{02} = 1225 \text{ m/s}$, $u_{02} = 0 \text{ m/s}$.

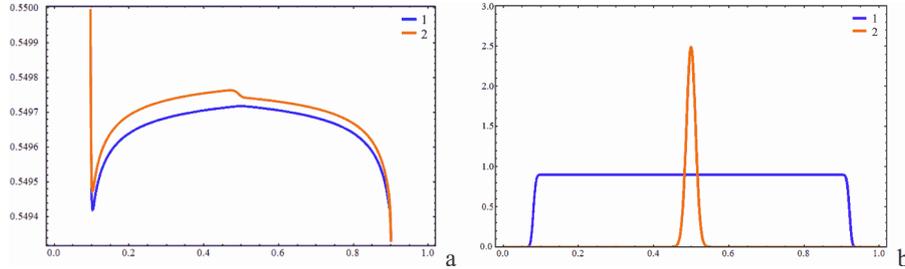


Figure 1. Profiles for pressure (a) and velocity (b) of second phase for Model 1 and Model 2

At the initial moment of time, the pressure gap in $\text{g}/(\text{cm} \cdot \text{s}^3)$ is set for each of the phases:

$$p_1(0, x) = \begin{cases} 9.0 \cdot 10^5, \\ 9.9 \cdot 10^5, \\ 9.0 \cdot 10^5, \end{cases} \quad p_2(0, x) = \begin{cases} 1.0 \cdot 10^5, & 0 \leq x < 0.5, \\ 1.1 \cdot 10^5, & 0.5 \leq x \leq 0.6, \\ 1.0 \cdot 10^5, & 0.5 < x \leq 1. \end{cases}$$

The calculation results are shown in Figures 2–7. The profiles of pressures and phase velocities are presented for different moments of time. Figures 5 and 7, representing the relative phase velocities, demonstrate the qualitative difference between the models in which the equilibrium of pressures in the phases is achieved even without such a condition. In the first case, there is no relative motion of the phases.

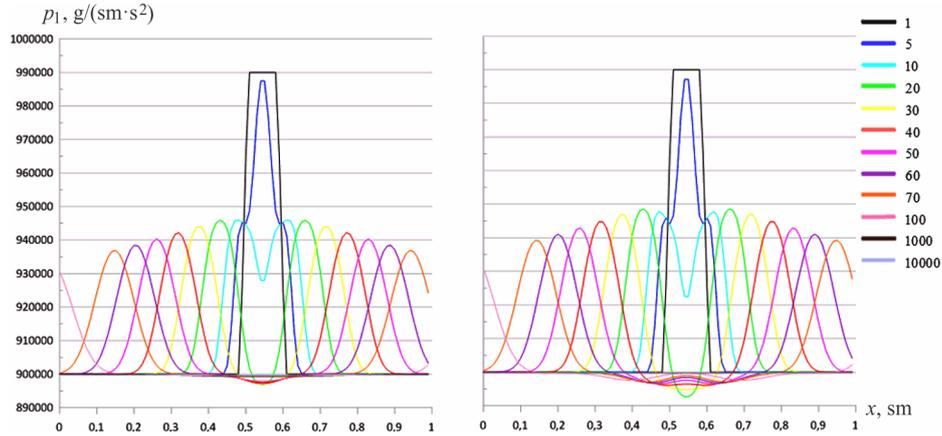


Figure 2. Pressure profiles in the first phase for Models 1 (left) and 2 (right) at different times

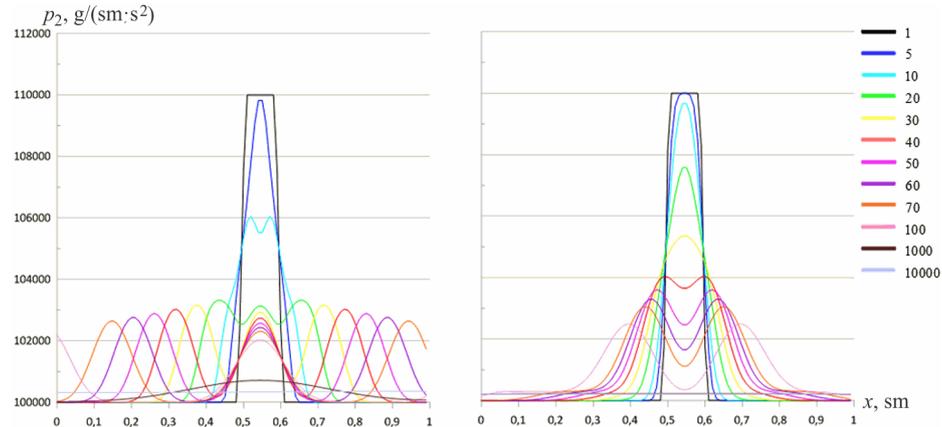


Figure 3. Pressure profiles for Models 1 (left) and 2 (right) at different times

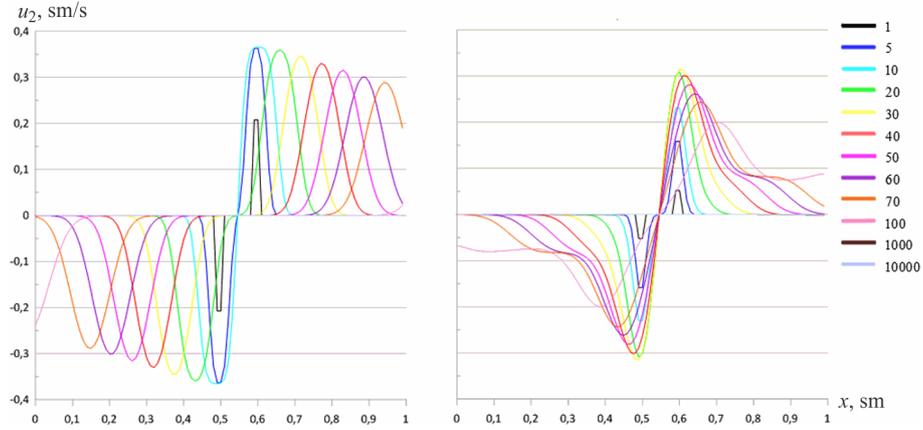


Figure 4. Velocity profiles of the second phase for Models 1 (left) and 2 (right) at different times

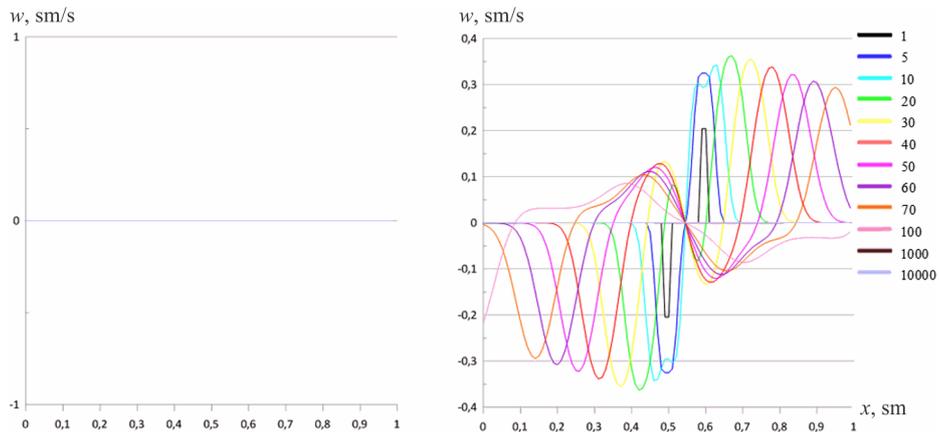


Figure 5. Relative velocity profiles of the second phase for Models 1 (left) and 2 (right) at different times

As follows from Figure 1, in these models there is no clearly defined compression wave front, as well as rarefaction waves. Therefore, a comparison was made with a two-velocity model of a two-phase medium with an explicit dependence of energy on the volume content of phases.

Model 3. In this model of hydrodynamics of a two-phase medium, obtained by the method of thermodynamically consistent systems of conservation laws [9], it is assumed that the phases are not in equilibrium with respect to pressure, as well as the dependence of the internal energy of a two-phase medium on the volume content of one of their phases α_1 . The dependence

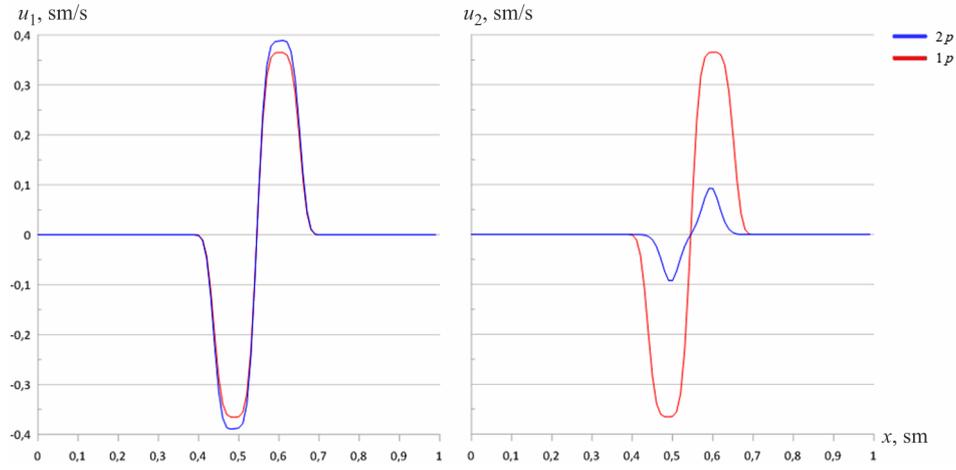


Figure 6. Velocity profiles of the first (left) and second (right) phases for Model 1 (1p) and Model 2 (2p) at different times

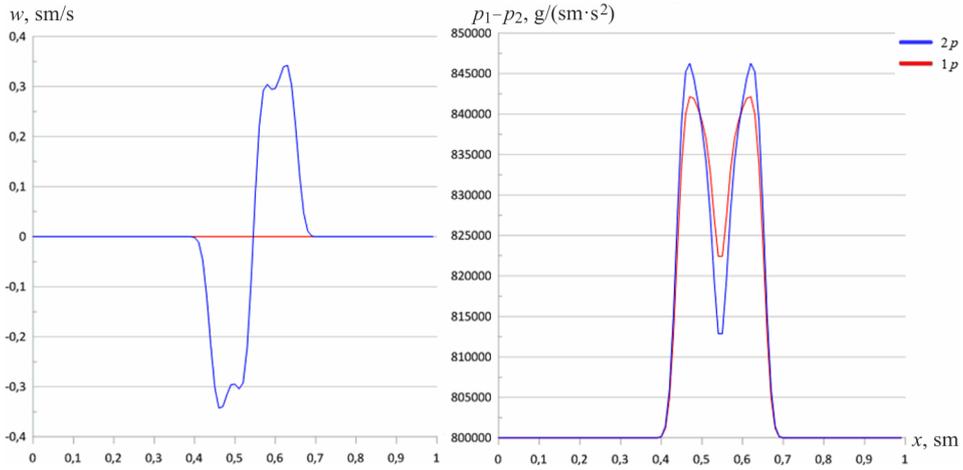


Figure 7. Profiles of the relative phase velocity (left) and the pressure difference in the phases (right) for Model 1 (1p) and Model 2 (2p) at the same time

of the internal energy (per unit volume) of such a two-phase medium will be as follows $E_0 = E_0(\rho, \rho_1, \alpha_1, S, \mathbf{j}_0)$

$$dE_0 = \mu d\rho + q d\rho_1 + (p_1 - p_2)d\alpha_1 + T dS + (\mathbf{w}, d\mathbf{j}_0)$$

The system of equations of motion of such a medium is supplemented by the equation for the transfer of the concentration of one of their phases [12]

$$\frac{\partial}{\partial t}(\rho\alpha_1) + \frac{\partial}{\partial x}(\rho u\alpha_1) = 0,$$

the equation of relative motion takes the form

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \left(\frac{1}{2} u_1^2 - \frac{1}{2} u_2^2 + h_1 - h_2 \right) + (\mathbf{u}, \nabla) \mathbf{w} - (\mathbf{w}, \nabla) \mathbf{u}_2 + b \mathbf{w} = 0.$$

Here $h_1 = e_1 + \alpha_1 \frac{p_1}{\rho_1}$, $h_2 = e_2 + \alpha_2 \frac{p_2}{\rho_2}$ is the enthalpy of phases, the energy of a two-phase medium is determined through the energies of the phases $E = \rho_1(e_1 + u_1^2/2) + \rho_2(e_2 + u_2^2/2)$.

The system of equations (7)–(10) is closed by setting the equation of state

$$p_1 = p(\rho_1, S), \quad p_2 = p(\rho_2, S), \quad T = T(\rho_1, \rho_2, S).$$

The result of calculations according to Model 1 and Model 2 is shown in Figure 8. A model of a two-phase medium with an explicit dependence of internal energy on the volume content of phases demonstrates the correct behavior of compression and rarefaction waves when a pressure jump is specified, which indicates the need to take into account such a dependence when studying acoustic and seismic phenomena in two-phase media and their interaction with hydrodynamic flows.

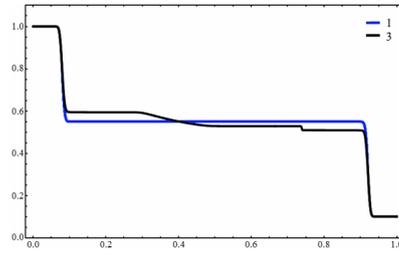


Figure 8. Velocity profiles of the second phase for Model 1 and Model 3 at the same time

Conclusion

In this work, the problem of discontinuity decay for thermodynamically consistent models of two-phase media of three types is numerically studied: equilibrium in pressure in phases, nonequilibrium in pressure in phases, and nonequilibrium in pressure with internal energy depending on the volume fractions of phases. The condition of non-equilibrium is decisive for the relative motion of phases when energy dissipation is taken into account only due to interfacial friction.

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