Bull. Nov. Comp. Center, Num. Model. in Atmosph., etc., 9 (2004), 51–56

Estimation of topography constraints on the numerical stability of a mountain wave simulation*

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Abstract. A three-dimensional nonhydrostatic meteorological model is applied to modeling of mountain waves. Numerical schemes with central differences in time and space on staggered grids are used at the so-called "adjustment" time integration stage. These schemes, being unconditionally stable for plane topography, are shown to be only conditionally stable for steep topography. The mountain steepness limitations necessary for numerical stability are obtained by solving numerically the amplification matrix eigenvalue problem.

1. Introduction

Mathematical hydrodynamical models are in wide use in meteorology to obtain a great amount of useful information in case of complex topography when few points of observation are not often representative for the actual climatic situation. In these models, the distribution of meteorological fields near the surface should be calculated with the highest possible accuracy. For this purpose, terrain-following coordinate transformations are widely used [1, 2]. It is of great interest to estimate the limitations of such transformations (an example of such estimation can be found in [3]).

A well-known approach to the numerical calculation of meteorological models, proposed by Marchuk [1], is the so-called "splitting" method. In this method, the solving of the original equation system is reduced to two main successive time steps: so-called "advection–diffusion" and "adjustment" stages. There is a variety of well-developed methods to solve the former stage equations. Most of the methods are based on the concept of monotonicity. The latter stage is usually more time-consuming and complicated to solve. These two main stages can be combined, thus giving a variety of splitting procedures.

In this paper, stability for the "adjustment" stage in three-dimensional formulation for numerical schemes with central differences on staggered grids is analyzed in order to obtain the necessary limitations on topography steepness that allow one to carry out stable simulations of mountain waves. Because of the high complexity of the three-dimensional system, the von

*This study is part of work supported by Fundamental Research Program of SB RAS under Grant 1.3.2 and the Russian Foundation for Basic Research under Grant 04-05-64562.

Neumann approach to stability analysis is used here, which gives necessary stability constraints [4].

In Section 2, the basic model equations are formulated. In Section 3, these equations are discretized and linearized. Section 4 is devoted to performing a von Neumann-type stability analysis and obtaining the necessary limitations on topography steepness. Discussion and conclusions are given in Section 5.

2. Basic equations

We consider here a small-scale nonhydrostatic model developed for simulations mainly in meso- and microscales (see, for example, [5]). In three-dimensional statement, the basic equations of motion, heat, moisture, and continuity in a terrain-following coordinate system are as follows:

$$
\begin{aligned}
&\frac{dU}{dt} + \frac{\partial P}{\partial x} + \frac{\partial (G^{13}P)}{\partial \eta} = f_1(V - V_g) - f_2 W + R_u,\\
&\frac{dV}{dt} + \frac{\partial P}{\partial y} + \frac{\partial (G^{23}P)}{\partial \eta} = -f_1(U - U_g) + R_v,\\
&\frac{dW}{dt} + \frac{1}{G^{1/2}}\frac{\partial P}{\partial \eta} + \frac{gP}{C_s^2} = f_2 U + g\frac{G^{1/2}\bar{\rho}\theta'}{\bar{\theta}} + R_w,\\
&\frac{d\theta}{dt} = R_\theta,\\
&\frac{ds}{dt} = R_s,\\
&\frac{1}{C_s^2}\frac{\partial P}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial}{\partial \eta}\left(G^{13}U + G^{23}V + \frac{1}{G^{1/2}}W\right) = \frac{\partial}{\partial t}\left(\frac{G^{1/2}\bar{\rho}\theta'}{\bar{\theta}}\right).
\end{aligned}
$$

In these equations, $U = \bar{\rho}G^{1/2}u$, $V = \bar{\rho}G^{1/2}v$, $W = \bar{\rho}G^{1/2}w$, $P = G^{1/2}p'$, where p', θ' are deviations from the basic state pressure $\bar{p}$ and potential temperature $\bar{\theta}$, s is the specific humidity, C_s is the sound wave speed, u_g, v_g are the components of geostrophic wind representing the synoptic part of the pressure, η is a terrain-following coordinate transformation:

$$
\eta = \frac{H(z - z_s)}{(H - z_s)},
$$

z_s is the surface height, H is the height of the top of the model domain. Here $H = \text{const}$,

$$
G^{1/2} = 1 - \frac{z_s}{H}, \quad G^{13} = \frac{1}{G^{1/2}}\left(\frac{\eta}{H} - 1\right)\frac{\partial z_s}{\partial x}, \quad G^{23} = \frac{1}{G^{1/2}}\left(\frac{\eta}{H} - 1\right)\frac{\partial z_s}{\partial y}.
$$

In the above equations, we use the following notation: for an arbitrary function φ

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \frac{\partial u\varphi}{\partial x} + \frac{\partial v\varphi}{\partial y} + \frac{\partial \omega\varphi}{\partial \eta} = \frac{\partial \varphi}{\partial t} + \mathrm{ADV}\varphi,$$

where

$$\omega = \frac{1}{G^{1/2}} w + G^{13} u + G^{23} v.$$

The terms R_u, R_v, R_ω, R_θ, R_s refer to subgrid-scale processes.

3. Linearized equations

Since the above three-dimensional equations are highly complicated, the von Neumann stability analysis approach is used here, which gives necessary stability constraints. In order to carry out a stability analysis of the von Neumann type, the basic equations system is linearized around a constant basic state wind velocity vector $(\bar{U}, \bar{V})$. Then the original equations are reduced to the following ones:

$$\begin{aligned}
&\frac{\partial U}{\partial t} + \frac{\partial P}{\partial x} = -\bar{U}\frac{\partial U}{\partial x} - \bar{V}\frac{\partial U}{\partial y} - \Delta G\frac{\partial P}{\partial \eta},\\
&\frac{\partial V}{\partial t} + \frac{\partial P}{\partial y} = -\bar{U}\frac{\partial V}{\partial x} - \bar{V}\frac{\partial V}{\partial y} - \Delta G\frac{\partial P}{\partial \eta},\\
&\frac{\partial W}{\partial t} + \frac{\partial P}{\partial \eta} = N\theta'' - \Delta H\frac{\partial P}{\partial \eta} - \bar{U}\frac{\partial W}{\partial x} - \bar{V}\frac{\partial W}{\partial y},\\
&\frac{\partial \theta''}{\partial t} = NW - \bar{U}\frac{\partial \theta''}{\partial x} - \bar{V}\frac{\partial \theta''}{\partial y},\\
&\frac{1}{C_s^2}\frac{\partial P}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial \eta} = -\Delta H\frac{\partial W}{\partial \eta} - \Delta G\frac{\partial U}{\partial \eta} - \Delta G\frac{\partial V}{\partial \eta}.
\end{aligned}$$

Here $\Delta G \sim G^{13} \sim G^{23}$, a measure of mountain steepness; $\Delta H \sim \left(\frac{1}{G^{1/2}} - 1\right)$, a measure of mountain height; $N^2 = \frac{g}{\theta}\frac{\partial \theta}{\partial z}$, the squared Brunt–Vaisala frequency; $\bar{U}$ and $\bar{V}$ are constant basic state wind velocity components; and $\theta'' = \frac{\rho'}{N}\frac{g\bar{\rho}}{\bar{\theta}}$.

4. Stability analysis

The two above systems of equations, the original and linearized ones, are discretized by using numerical schemes with central differences in time and space, on grids for the scalar and vector quantities shifted half-grid size from each other in all three space variables (see, for example, [5]). The terms in the left-hand side of the linearized system are taken by central differences in time and space, while the terms in the right-hand side are taken at half-time grid levels [6]. In the von Neumann stability analysis procedure, one needs to estimate the amplification factor of the total grid operator. Because of the high complexity of the linearized equations, this is not a simple task, and one has to perform some simplifications.

Since stability is studied only at the adjustment stage, we put $\bar{U}$,$\bar{V}$, and $\bar{N}$ equal to zero. In paper [6], a two-dimensional (x, z) stability analysis was carried out. In this case, it was possible to obtain the following characteristic equation:

$$\left[1+\frac{X}{4}\right]\lambda^4 + (C_s^2\Delta t)^2[kx^*kz^{**}\Delta G + kz^{*2}\Delta H]\lambda^3 +$$
$$2\left[-1+\frac{X}{4}+\frac{(\Delta Gkz^{**})^2+(\Delta Hkz^*)^2}{2}(C_s^2\Delta t)^2\right]\lambda^2 +$$
$$(C_s^2\Delta t)^2[kx^*kz^{**})\Delta G + kz^{*2}\Delta H]\lambda + \left[1+\frac{X}{4}\right] = 0.$$

Here $X = (C_s^2\Delta t)^2[kx^{*2}kz^{*2}]$,

$$kx^* = \frac{2\sin(kx\Delta x/2)}{\Delta x}, \qquad kz^* = \frac{\sin(kz\Delta\eta/2)}{\Delta\eta},$$
$$kz^{**} = \frac{\sin(kz\Delta\eta)\cos(kx\Delta x/2)}{\Delta\eta},$$

kx and kz are horizontal and vertical wave numbers, respectively. This equation was solved analytically by Ferrari's method [6].

At the "adjustment" stage, the total three-dimensional difference equations system may be written down as follows:

$$(A+\Delta tC)S^{n+1} = (A-\Delta tC)S^{n-1} + BS^n,$$

where $S^n = (P^n, U^n, V^n, W^n, \theta''^{\,n})'$, and the matrices A, C, and B are as follows:

$$A = \begin{vmatrix} 1/C_2^s & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \quad C = \begin{vmatrix} 0 & ikx^* & iky^* & ikz^* & 0 \\ ikx^* & 0 & 0 & 0 & 0 \\ iky^* & 0 & 0 & 0 & 0 \\ ikz^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$B = 2\delta t\begin{vmatrix} 0 & -\Delta Gikz^{**} & -\Delta Gikz^{**} & -\Delta Hikz^* & 0 \\ -\Delta Gikz^{**} & 0 & 0 & 0 & 0 \\ -\Delta Gikz^{**} & 0 & 0 & 0 & 0 \\ -\Delta Hikz^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

where, in addition, $ky^* = \dfrac{2\sin(ky\Delta y/2)}{\Delta y}$.

In the three-dimensional case, it is not an easy task to obtain an analytical solution. Instead of calculating the characteristic equation, the eigenvalue problem for the amplification matrix is solved by using a procedure for matrices in the Hessenberg form described by Wilkinson and Reinsch [7] (see also [8]). The input parameters are used as in [6]: $(\Delta x, \Delta y, \Delta\eta, C_s) =$ (1200 m, 1200 m, 200 m, 340 m/s). At $\Delta t = 12$ s, we have found instability for any ΔG. Reducing Δt to 2 s, the calculations have shown that, similar to the two-dimensional case considered in [6], the necessary stability limitation on ΔG is as follows:

$$0 \leq \Delta G \leq \gamma < 1,$$

where γ is about 0.25.

5. Conclusion

The purpose of this paper was to carry out a three-dimensional linear stability analysis in order to obtain the necessary limitations on topography steepness. The above results have shown that the limitations remain generally the same as those obtained in a two-dimensional linear stability analysis performed by an analytical method in [6]. The central-difference-type schemes used in the model are universally known as unconditionally stable for plane topography. The analysis performed above has shown that the researcher should be careful in applying such schemes to cases of steep topography.

References

[1] Marchuk G.I. Numerical Methods in Weather Prediction. — Academic Press, 1974.

[2] Penenko V.V., Aloyan A.E. Models and Methods for Environmental Problems. — Novosibirsk: Nauka, 1985 (in Russian).

[3] Yudin M.S. Coordinate transformation for a small- scale meteorological model // NCC Bulletin, Series Num. Model. in Atmosph. — Novosibirsk: NCC Publisher, 2002. — Issue 2. — P. 101–107.

[4] Richtmyer R.D., Morton K.W. Difference Methods for Initial-Value Problems: 2nd ed. — New York: Wiley, 1967.

[5] Yudin M.S. A splitting algorithm for a compressible atmospheric model // Methods of Mathematical Modelling for Problems of Environmental Protection and Ecology / V.V. Penenko ed. — Novosibirsk: Computing Center, 1991 (in Russian).

[6] Ikawa M. Comparison of some schemes for nonhydrostatic models with orography // J. Meteor. Soc. Japan. — 1988. — Vol. 66, No. 5. — P. 753–776.

[7] Wilkinson J.H., Reinsch C. Handbook for Automatic Computation, Linear Algebra II. — New York: Springer, 1971.

[8] Press W.H., Flannery B.P., Teukolsky S.A., Vetterling W.T. Numerical Recipes – the Art of Scientific Computing. — Cambridge: Cambridge Univ. Press, 1986.