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Research into periodic signal processing by the median filters

Vladimir I. Znak

Abstract. We discuss a specific nature of the phase-synchronized-weighted (Synphase-Weighted) Median Filters when processing frequency-modulated (possibly, swip) signals recorded at a discrete time. It is shown that the response of the Synphase-Weighted Median Filters weaker depends on the filter length than the response of conventional median filters. The conclusions obtained are confirmed by the results of numerical statistical modeling. The algorithm of processing of a signal with a frequency band 5 Hz is presented.

1. Introduction

The median filters are of interest to us as means of preserving signal steps, sharp discontinuities and edges when processing a signal for the noise suppression. The nature of a median and principles of appropriate filtering are very simple. A median is the central value of numbers sorted in the ascending order, and the median filtering of any sequence is the replacement of the central element (CE) of the sequence by the median. In this way, the median filtration for the sequence

$$\{y_1, \dots, y_n\} = Y, \tag{1}$$

is the following:

$$u_{0.5(n)} = \mathrm{MED}_n(y_{c-\nu}, \dots, y_c, \dots, y_{c+\nu}) = \tilde{y}_c, \tag{2}$$

where $\tilde{y}_c$ is the central value of the variational row

$$\{\tilde{y}_{c-\nu} \le \dots \le \tilde{y}_c \le \dots \le \tilde{y}_{c+\nu}\} = \tilde{Y}, \quad c \in Z, \tag{3}$$

composed of the terms of sequence (1). Here $n = 2\nu + 1$ is odd.

Thus, we consider the median filter as a particular case of the percentile filter

$$u_{\alpha(n)} = \mathrm{RANK}_{r(n)}(y_{c-\nu}, \dots, y_r, \dots, y_{c+\nu}) = \tilde{y}_r,$$

where $\alpha = (r-1)/(n-1)$ is measure of the size of the corresponding sample of the sequence Y. For a median, we have $\alpha = 0.5$, $r = c$.

Let to each $y_i \in Y$ a rational number or the weight $w_i \in W$, $i = \overline{1, n}$, be assigned, where the weight w_i is interpreted as the number of copies of the element y_i (usually, weights are selected symmetric with respect to the CE of sequence (1)). In this connection, $N = \sum_{i=1}^{n} w_i$ is the length of the sequence. Now, to row (3) we assign the sequence of weights $\tilde{W} = w_j(i)$, $i, j = \overline{1, n}$, where $w_j(i)$ are rearranged to permutations of the elements $y_i \in Y$ according to their rank. A median is selected by the consecutive account of the weights up to the time of fulfilling the condition $\sum_{j=1}^{c} w_j(i) = (N+1)/2$:

$$u^w_{0.5(n)} = \mathrm{MED}_{n(w)}\big(y^{(w(c-\nu))}_{c-\nu}, \ldots, y^{(w(c))}_{c}, \ldots, y^{(w(c+\nu))}_{c+\nu}\big) = \tilde{y}^{(w(i))}_{c}. \tag{4}$$

The non-integer weights can be transformed to integers by introduction of an appropriate factor. Schemes for operation with non-integer weights are offered in paper [1]. Some specific features of the median filters were determined in works [2, 3]. In the research into the filter response to a cosine function [2], the author comes to the conclusion about likelihood of the spectral characteristic of the moving medians and the moving averaging. The result is presented for a simple unweighted filter provided that the length of the filter does not exceed the period of the function. In paper [3], the authors propose a modified trimmed mean (MTM) filter containing the features both of the median and the linear filters. But the authors do not investigate the behavior of this filter on periodic signals.

Our task is to propose such a modification of a median filter that would be adapted for the processing of a frequency-modulated signal, to investigate the behavior of this filter, and to weaken the dependence of the filter response on length.

2. Basic definitions

In our case, we consider a signal, recorded in the digital mode during some discrete time using a preassigned time interval $\Delta t = const$. First of all it is worthwhile to dwell in more detail on the basic definitions.

2.1. The case of a harmonic signal. Let us address to a sinusoidal harmonic function

$$y(t) = \sin(\varphi + \varpi t), \tag{5}$$

recorded at discrete moments of time as the sequence

$$Y = \{y_i = \sin(\varphi + \varpi i\Delta t),\ \ i = 0, \pm 1, \ldots, \pm\nu\}, \tag{6}$$

where $n = 2\nu + 1 = 2RT$ is the length of the record and T is the period of the function. Let us assume the length of the sequence be such that the condition

$$(\nu - 1)\varpi\Delta t \le R2\pi \le \nu\varpi\Delta t \tag{7}$$

is satisfied.

Then

$$V_R = \frac{RT}{\Delta t} = \frac{R2\pi}{\varpi\Delta} \tag{8}$$

is the evaluation of R periods of the signal in quanta of time ($R \geq 1$ is integer).

Let to elements $y_i \in Y$ of sequence (6) there be assigned weights such that $w_c = w_{c\pm jT} = 1$, $j = \overline{1, R}$, while the other weights are equal to zero. The filter with these features will be called a co-phased (phase-synchronized-weighted) or synphase weighted (SPhW) filter. The weight of the central element can exceed the weights of other components — for emphasizing its value — more than by unit. It is the case of the weighted SPhW filter. The computation of the corresponding weights and the response of the SPhW filter to a harmonic signal are established in paper [4]. Because of importance of the algorithm of calculating weights for the co-phased components of the sequence Y, let us present here:

1. Computation of length of the filter as

$$\lceil n \rceil, \ \text{where } n = \frac{2RT}{\Delta t} + 1; \tag{9}$$

2. Realization of assignments: $w_0 = 1$, $w_{\pm i} = 0$, $i = \overline{1, \nu}$;
3. Introduction of the integer variable $j = 1, \ldots, R$, where j is a cycle index;
4. Computation of indices of the elements with nonzero weights, and computation of the corresponding weights according to the rules:

   i) $V_j = jT/\Delta t$;
   ii) if $V_j = \lceil V_j \rceil = \lfloor V_j \rfloor$, then $w_{\pm V_j} = 1$ and go to Item 5; else
   iii) $w_{\pm\lceil V_j \rceil} = V_j - \lfloor V_j \rfloor$; $\quad w_{\pm\lfloor V_j \rfloor} = \lceil V_j \rceil - V_j$;

5. If $j < R$, then $j := j + 1$ and go to Item 4i; else exit from the loop and finish the procedure.

Here, we designated as $\lceil x \rceil$ the smallest integer greater or equal to x (the nearest from above) and as $\lfloor x \rfloor$ — the largest integer less or equal to x (the nearest from below).

2.2. The case of a frequency-modulated signal. The response of the SPhW filter to a frequency-modulated signal is established in paper [5]. In this paper, based on the results of paper [5] we will demonstrate, with attraction of a specific example, the weakening of dependence of the response of the SPhW filter on length, and propose the algorithm of processing of a signal with frequency band 5 Hz.

Now, we turn to the sequence of sinusoidal functions

$$\{y_l^{(\mathrm{sg})}(t),\ \ l=\overline{1,m}\},\ \ \text{where}\ \ y_l^{(\mathrm{sg})}(t)=\sin(\varphi_l+\varpi_l^{(\mathrm{sg})}t), \tag{10}$$

recorded at discrete moments of time $t_{i(l)}$ with a permanent period of quantization Δt and with the circular frequencies $\varpi_l^{(\mathrm{sg})}$ (the frequency band Δf of the sequence is constrained: $\Delta f = f_m - f_1$). Such a sequence is a frequency-modulated (FM) signal. We will present a sample of the l-th term of sequence (10) as a row

$$y_i^{(\mathrm{sg})}(l)=\sin(\varphi_l+\varpi_l^{(\mathrm{sg})}i\Delta t),\ \ i=0,\pm1,\ldots,\pm\nu \tag{11}$$

of the length $n = 2\nu+1$ satisfying conditions (7), (8). Let $T^{(\mathrm{fl})}$ be the evaluated (own) period of the SPhW filter and $T^{(\mathrm{fl})} \sim T_i^{(\mathrm{sg})}$. The response of the SPhW filter of length $N = 3$ $(R = 1)$ is the following:

$$u_{0.5,n(\phi_1)}^{f,w}(l)=\mathrm{MED}_{n(w)}\big(\sin\varphi_l,\sin(\varphi_l\pm 2\pi(T_l^{(\mathrm{sg})}+\varsigma_l)/T_l^{(\mathrm{sg})})\big). \tag{12}$$

Here $\varsigma_l = T^{(\mathrm{fl})} - T_l^{(\mathrm{sg})}$. So, the estimation of the relative period deviation is

$$\Delta_l=\frac{f_l^{(\mathrm{sg})}}{f^{(\mathrm{fl})}}-\left[\frac{f_l^{(\mathrm{sg})}}{f^{(\mathrm{fl})}}\right], \tag{13}$$

where $[x]$ is the integer nearest to x.

In a more general case, expression (13) acquires the form

$$\Delta_{l,j}=\frac{jf_l^{(\mathrm{sg})}}{f^{(\mathrm{fl})}}-\left[\frac{jf_l^{(\mathrm{sg})}}{f^{(\mathrm{fl})}}\right]. \tag{14}$$

Here $-0.5 \le \Delta_{l,j} \le 0.5$ or

$$|\Delta_{l,j}| \le 0.5. \tag{15}$$

Now the response (12) of the SPhW filter will look like:

$$u_{0.5,n(\phi_1)}^{f,w}(l)=\mathrm{MED}_{n(w)}\big(\sin\varphi_l,\sin(\varphi_l\pm\Delta\varphi_l)\big), \tag{16}$$

where $\Delta\varphi_l = 2\pi(T_l^{(\mathrm{sg})}+\varsigma_l)/T_l^{(\mathrm{sg})}$.

Let

$$Q_l=\int_0^{2\pi}\frac{u_{0.5,n(\phi_1)}^{(f,w)}(l)}{y_l}\,d\varphi_l,\qquad 1\le l\le m, \tag{17}$$

be the generalized estimation of the response quality of the SPhW filter, where $y_l = \sin\varphi_l$ is the signal value corresponding to the CE of sequence (11). It is possible to demonstrate that

$$\frac{Q_l}{2\pi}\cong 1-\frac{|2\Delta\varphi_l|}{\pi}. \tag{18}$$

The behavior of this estimation as function of $|\Delta_{l,j}|$ is shown in Figure 1.

Now for the average estimation

$$\overline{\Delta}_l = \frac{1}{m}\sum_{l=0}^{m}\left(\frac{jf_l^{(\mathrm{sg})}}{f^{(\mathrm{fl})}} - \left[\frac{jf_l^{(\mathrm{sg})}}{f^{(\mathrm{fl})}}\right]\right)$$

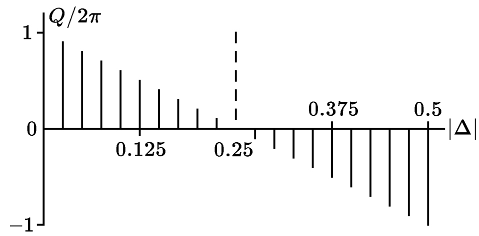

Figure 1. Generalized estimation of the response quality

and under the condition of the own filter frequency band

$$\Delta f^{(\mathrm{fl})} = \{f_1^{(\mathrm{fl})}, f_2^{(\mathrm{fl})}, \ldots, f_k^{(\mathrm{fl})}\},$$

we have the functional

$$\overline{\Delta}_{r,l} = \frac{1}{m}\sum_{l=0}^{m}\left(\frac{jf_l^{(\mathrm{sg})}}{f_r^{(\mathrm{fl})}} - \left[\frac{jf_l^{(\mathrm{sg})}}{f_r^{(\mathrm{fl})}}\right]\right). \tag{19}$$

The results are obtained under the condition of equality of the central element weight to unit. If the weight of the CE exceeds the weights of other components, then the results obtained are applicable in this case, too, because the correlation of ranges of the variable φ_l and the function $\Delta_{l,j}$ is saved. Emphasizing the value of the central element will only improve the quality of such a response as this will displace a centroid to φ_l. However, the nonlinear character of the functions such as median and trigonometrical essentially restricts the possibility of their analysis and obtaining more exact estimations.

From the analysis of expressions (18), (19), we can turn to the problem of selecting the structure of the SPhW filter oriented towards a frequency-modulated signal processing.

3. Comparison of the SPhW and conventional median filters

Let us compare the degree of preservation of a periodic signal on the output of the SPhW filter to that on the output of the corresponding conventional median filter.

Expressions (18), (19) make it possible to conclude that the SPhW filter with a length $N = \{3, 5\}$ together with $f^{(\mathrm{fl})} = 7$ Hz, will save a satisfactory quality of a signal in the frequency band $6.3 \div 7.75$ Hz.

Let us have a record of this signal with the quantization interval equal to $\Delta t = 0.02$ s, and a frequency step 0.1 Hz. The period (time of existence) of the model is 40 s; and the time of existence of each frequency is the same. We will use the autocorrelation coefficient $K(y, u)$, where y is the input of

Table 1. Autocorrelation coefficients for unweighted filters

n	Med	SPhW
3	0.99	0.99
5	0.93	0.99
7	0.27	0.98
9	−0.90	0.93
11	−0.92	0.84

Table 2. Autocorrelation coefficients for weighted filters

n/w	Med	SPhW
3/3	0.99	1.00
5/3	0.99	0.99
7/5	0.93	0.99
9/5	0.76	0.99
11/7	0.27	0.96
13/7	0.37	0.95
15/9	0.37	0.95

the filter, u is the output, in order that the degree of preservation of a signal on the output of the filters be estimated. Results of estimations are given in Table 1 (the results of testing the unweighted filters), and Table 2 (the results of testing the weighted filters). In Table 1, the column n includes the filter lengths, and Med, SPhW are autocorrelation coefficients for the conventional median and the co-phased filters, respectively. Table 2, in addition to the filter lengths, includes the weight of the central element—w (after slash, i.e., n/w denotes the length n and the weight of the CE). The appropriate dependence of the signal preservation degree on lengths are shown in Figure 2, for unweighted, and in Figure 3, for weighted filters, respectively.

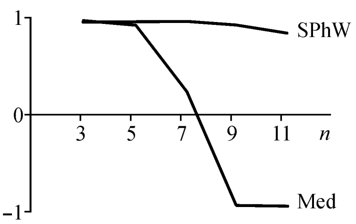

Figure 2. Autocorrelation coefficients for unweighted filters

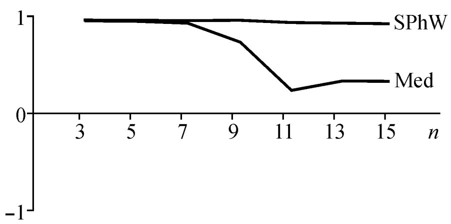

Figure 3. Autocorrelation coefficients for weighted filters

In principle, this estimation of the behavior of the SPhF filter is valid for an optional sample of $f^{(\mathrm{fl})}$ and the correspond frequency band. In this case, it is important to satisfy the condition of the ratio between the repetition rate of time sampling and $f^{(\mathrm{fl})}$.

4. The parallelizing as a principle of expansion of frequency band of the SPhW filter

In the previous section, we have demonstrated the process of working up a frequency-modulated signal with the frequency band $\Delta f \approx 1.5$ Hz. Here, we will demonstrate that there exists a possibility of signal processing with a wider frequency band. At the same time, we will show restoring a signal from noise. For this purpose, we will show the parallel organized structure of filter that permits signals processing in a wide frequency band. The one version of such structure of the SPhW parallel organized filter is offered in [6]. The threshold principle of the composition/restoration of a signal filtered by a n-tap structure SPhW filter was used there. The median principle of a signal restoration is concerned here.

For the illustrative purposes, we consider the processing of a linear frequency-modulated (LFM) signal using the numerical statistical modeling. Let the source signal be LFM signal with a frequency band 5–10 Hz and with a frequency step equal to 0.1 Hz. The quantization interval is $\Delta t = 0.02$ s. The period (time of existence) of the signal is 40 s. For distortion of a signal, white noise with zero mean Gaussian distribution is used. The level of the noise is such that a signal value is five times lower than the level of noise (on the average).

For the signal processing in our case, we can use the algorithm, whose flow-charts are shown in Figures 4, 5. The algorithm includes three parallel directed sets (Set 1, Set 2, Set 3; Figure 4), each directed set including eight parallel lines (Figure 5). At first, we consider functions of operators of these lines.

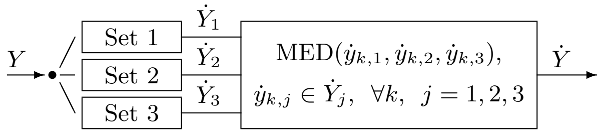

Figure 4. Flow-chart of signal processing

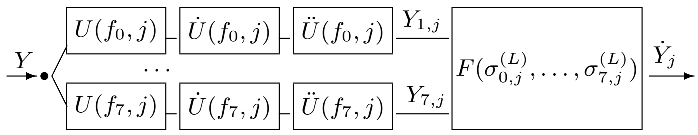

Figure 5. Signal processing on Set j, $j = 1, 2, 3$

First of all, operators of each i-th line ($i = 0, \ldots, 7$) are calculated for own frequency: $f_0 = 5.15$ Hz, $f_1 = 5.85$ Hz, $f_2 = 6.5$ Hz, $f_3 = 7.1$ Hz, $f_4 = 7.7$ Hz, $f_5 = 8.25$ Hz, $f_6 = 8.9$ Hz, $f_7 = 9.55$ Hz. Here the weights are calculated with the use of the above-described algorithm.

The operators $U(f_i, j)$ of one separate Set j, $j = 1, 2, 3$, are identical to each other with the exception of the own frequency f_i. The same remark is valid for the rest operators $\dot{U}(f_i, j)$, $\ddot{U}(f_i, j)$. As far as distinctions of these operators for separate directed sets are concerned, they are: $U(f_i, 0)$ implements the function $u^{f_1,3}_{0.66,N(\phi_i} + u^{f_i,3}_{0.34,N(\phi_1)}$; $\dot{U}(f_i, 0)$ implements $u^{f_i,3}_{0.71,N(\phi_2)} + u^{f_i,3}_{0.3,N(\phi_2)}$; $\ddot{U}(f_i, 0)$ implements $u^{f_i,3}_{0.5,N(\phi_2,\phi_3)}$.

The operator $F(\sigma_{0,j}^{(L)}, \dots, \sigma_{7,j}^{(L)})$ reconstructs the LFM signal in all the frequency band. As basis of the operator for building-up the signal we use a standard deviation of a signal with some basis L of the signal values:

$$\sigma^{(L)} = \left(\sum_{i \in L} (x_i - M(x))^2 \right)^{1/2}. \tag{20}$$

Let the basis L be running along the signal values from the beginning to the end of the record, for each line. Hence, we will have a set of standard deviations $\Theta_j^{(L)}(k) = \{\sigma_{0,j}^{(L)}(k), \dots, \sigma_{7,j}^{(L)}(k)\}$ for each k-th moment of the processing. The examined operator picks out as $\dot{y}(k) \in \dot{Y}_j$ such $y_{i,j}(k) \in Y_{i,j}$, $i = 0, \dots, 7$, $j = 1, 2, 3$, for which corresponds a maximum value of a standard deviation, i.e, the function

$$f(\sigma_{i,j}^{(L)}(k), y_{i,j}(k)) = \dot{y}(k) \quad \text{if } \sigma_{r,j}^{(L)}(k) \geq \sigma_{i,j}^{(L)}(k) \tag{21}$$

is realized for any $\sigma_{i,j}^{(L)}(k), \sigma_{r,j}^{(L)}(k) \in \Theta_j^{(L)}(k)$.

Three iterations of the signal processing for $L = 9$ were realized: the output $\dot{Y}$ was used as repeated input of the algorithm.

The results of the test data of the statistical trial are given in Table 3. The iterations of the processing are shown in table rows: $K_I^{(*)}/K_I^{(0)}$, ..., $K_{III}^{(*)}/K_{III}^{(0)}$ (slash is used to denote first/second lines of the table). Here, the first lines of the table contain the autocorrelation coefficients for a noise signal and the second lines—the autocorrelation coefficients for a pure signal (not deformed by noise). At the same time, $K_0^{(*)}$, $K_0^{(0)}$ are the autocorrelation coefficients before processing. The degree of preservation of a pure signal on the output of the filter after iterations is shown in Figure 6. The

Table 3. Autocorrelation coefficients for frequency-modulated signals: $K_0^{(*)} = 0.4235$, $K_0^{(0)} = 1.0$

Coefficient	Directed set			MED
	1	2	3	
$K_I^{(*)}$	0.5	0.4236	0.562	0.5342
$K_I^{(0)}$	0.998	0.989	0.967	0.992
$K_{II}^{(*)}$	0.5840	0.5505	0.6403	0.6076
$K_{II}^{(0)}$	0.99	0.987	0.963	0.987
$K_{III}^{(*)}$	0.6253	0.6089	0.6486	0.6329
$K_{III}^{(0)}$	0.986	0.984	0.96	0.984

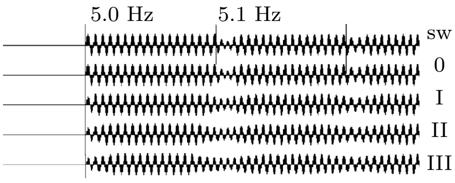

Figure 6. Change of the waveform of a pure signal on the output of the filter after iterations of processing

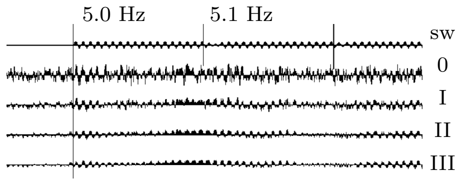

Figure 7. Restoration of noise signal on the output of the filter after iterations of processing

degree of restoration of a noise signal on the output of the filter after iterations is shown in Figure 7. The length of the signal does enable not us to present it in full volume in the figures, therefore here only the lower frequency are shown (5.0 Hz, 5.1 Hz).

One can judge about the degree of preservation of a pure signal and about the degree of restoration of a noise signal.

Finally, the above-described algorithm does not pretend to be the optimal algorithm (i.e., the one having a minimum number of computing operations used for the maximum precision). This problem demands separate attention. Our purpose was to demonstrate the existence of a possibility of processing of the FM signals with a wide frequency band by the SPhW filters.

5. Conclusion

Thus, in any case of processing of a periodic signal, the behavior of a co-phased median filter is more preferable against the background of the behavior of a conventional median filter in all cases without exception (see Figures 2 and 3).

Let us make one remark concerning the signal processing by such filters: variation of a rank up/down in regard to a median opens up a possibility for super-position of outputs of a pair of filters working in parallel, such that one of them is a percentile filter $u_{\alpha,N(\phi_1,\ldots,\phi_R)} = \mathrm{RANK}^{\alpha}_{N(w)}(y_0, \phi_V, \ldots, \phi_{\pm RV})$, and the other is $u_{\beta,N(\phi_1,\ldots,\phi_R)} = \mathrm{RANK}^{\beta}_{N(w)}(y_0, \phi_V, \ldots, \phi_{\pm RV})$. At the same time, the equality $(\alpha + \beta)/2 = 0.5$ should be fulfilled. The design of such a filter is close to the idea of the Modified Trimmed Mean Filters stated in [3]. However, in our case, the use of such a composite filter demands some attention: the filter is transformed to the category of the linear averaging ones if the parameters α, β exceed certain values (this is manifested in the effect of smoothing a step-like function).

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